Year 12 Maths Extension 2

Lesson 7

Introduction to Proof

MATRIX EDUCATION

1. Overview of Proofs

☐ Introduction to Proof

- Mathematics is the study of reason and logic, and central to this is being able to string together a coherent argument to *prove a statement.*
- Proof forms an important part of the new HSC Extension 2 syllabus, and if you do any maths at university, you will be exposed to many more proofs as well.
- Basic examples of proofs we have already seen include those in complex numbers and polynomials (e.g. the remainder theorem) and geometry (e.g. Pythagoras' Theorem). This is not a new concept to you, but we will discuss proofs in much further depth in this unit.
- In this lesson, we will describe the basic language underpinning proofs, including *statements*, *implications,* and *conditional statements*. We will also define the logical relationship between statements, including the *converse, negation* and *contrapositive*.
- Finally, we will define what a proof is, give some basic examples of proofs, and show how we can use counterexamples to show that a statement is false (a *disproof).* You will learn how to write proofs more rigorously and readably, as well as be able to identify and avoid the common mistakes that occur in the writing of proofs.
- While the syllabus has an emphasis on induction, inequalities and number theory – it is not unfeasible that there will be guided questions in the HSC involving proofs involving other topics from Extension 1 and Extension 2. In fact, this was always implicitly the case in most years before 'Proof' was even part of the syllabus.

2. Statements and Implications

- In this section, we will learn the basic terminology underpinning logic and proof.
- We start with the most basic terminology of all: that of a *statement*.

☐ Introduction to Statements

- Mathematics and logic is concerned with *truth*: we want to prove that sentences are either definitely true or definitely not true.
- Such a sentence is called a **statement**.

A **statement** is a sentence that is either definitely true, or definitely false.

- The following are all examples of true statements:
  - $1 + 1 = 2$
  - 3 is an integer.
  - The area of a circle with radius r is πr^2.
  - Sydney is in New South Wales.
- The following are all examples of false statements:
  - 17 = 3.
  - All triangles have a right angle.
  - All polynomials have a real root.
- The following are not examples of statements, since it doesn't make sense to talk about truth:
  - 17
  - Multiply by 2.
  - What is the solution of $x + 2 = 3$?
- Statements should be definitely true or definitely false, *without* needing extra information. For instance, the following are not statements:
  - *My teacher is male.* This is could be true or false, depending on your teacher.
  - $2x = 84$. This is true if x happens to equal 42, but otherwise it's false.

Concept Check 2.1

State whether or not the following sentences are statements. If they are statements, say if they are true or false, if possible. (This is called the ***truth value*** of the statement).

(i) Bubble tea. [1] **1**

(ii) It is raining today. [2] **1**

(iii) If a, b and c are the sides of a triangle, then $a^2 + b^2 = c^2$. [3] **1**

(iv) If a, b and c are the sides of any triangle, then $a^2 + b^2 = c^2$. [4] **1**

(v) $x = 1$. [5] **1**

(vi) There is an integer x, such that $x = 1$. [6] **1**

(vii) If $x = 1$, then $x^2 = 1$. [7] **1**

(viii) All even integers greater than 2 are the sum of two prime numbers. [8] **1**

(ix) This sentence is false. [9] **2**

Did you know?

This is called the Liar's Paradox and has a great history in formal logic! You can read about it here:

https://en.wikipedia.org/wiki/Liar_paradox

Combining Statements

- We can use the words **and** & **or** to combine two statements, to give another statement.
- For example, consider the following two statements:

A) The number 2 is even.

B) The number 3 is even.

- We can combine them to give two new statements:

*C) The number 2 is even **and** the number 3 is even.*

*D) The number 2 is even **or** the number 3 is even.*

- Common sense allows us to state the truth values of these new statements:

Statement C is ________[10], statement D is ________[11]

An **and** statement is true if ***both*** component statements are true.

An **or** statement is true if ***one or both*** of the component statements are true.

Negating Statements

- Another way of obtaining a new statement from another statement P is to form the statement **it is not true that P.**
  - This is called the **negation** of the statement.
  - This is often written ***Not P***.
- For example, if P is the statement

2 is an even number

then the negation $Not\ P$ is the statement

It is not true that *2 is an even number.*

  - Is this negation a true statement? ________[12]
  - What's a more succinct way of writing this negation statement? [13]

____

Not P is true if ***P*** is false, and vice versa.

We **reverse the truth value.**

Concept Check 2.2

Negate the following statements:

(a) Hugh is an expert in Extension 2 Mathematics. [14] **1**

(b) Every dog has a name. [15] **1**

(c) Nigel and Callum are Chatswood maths tutors. [16] **1**

(d) $f(x) \geq 100$ for all x in the domain of $f(x)$. [17] **1**