Year 12
Maths
Extension 2

Lesson 3
Harder Motion 1

MATRIX
EDUCATION

3. Application Of Integration To Motion Formulae

☐ Finding the Displacement Given Velocity

- Given the velocity function of a particle, the displacement function is found by integration:

$$\int v(t)\ dt = x(t) + C$$

Note to students

The constant C is found by using the initial conditions.

- Or, if we only want to find the change in displacement, we can use a definite integral to find:

$$\text{Change in Displacement from } t_1 \text{ to } t_2 = \int_{t_1}^{t_2} v(t)\ dt$$

☐ Finding the Distance travelled

- The distance travelled can be found by considering the **area** between the curve and the horizontal axis.
- Remember that, when considering areas; we **need to consider** if the velocity graph is positive or negative.
- When asked to find the distance travelled, **always draw a sketch!** This will help us evaluate when we have positive or negative areas.
- Generally:

$$\text{Distance Travelled} = (\text{Positive Areas}) + |(\text{Negative Areas})|$$

Example

A particle is moving on the x –axis and is initially at the origin. Its velocity, v ms[-1], at time t seconds is given by

$$v = \frac{t}{9 + t^2}$$

(a) Find the initial velocity of the particle. [1] 1

Find v when $t = 0$.

(b) The distance the particle travelled over the first 5 seconds. [2] 2

The graph $v = \frac{t}{9+t^2}$ is only negative when t is negative; so we do not need to worry about a sketch, and just use standard integration.

Evaluate $x = \int_{t_1}^{t_2} v(t)\, dt$ when $t_2 = 5$ and $t_1 = 0$.

Concept Check 3.1

Particles A_1 and A_2 move with velocities $v_1 = 6 + 2t$ and $v_2 = 4 - 2t$, in terms of metres and seconds. Initially both particles are at $x_1 = 2$ and $x_2 = 1$ respectively.

(a) Find the expression for x_1 and x_2.[3] 3

(b) Prove that the particles never meet and determine the minimum distance between them. [4] 2

(c) Prove that the midpoint M between the two particles is moving with constant velocity and find its distance from each particle after 3 seconds. [5] 3

☐ Finding the Velocity, Given Acceleration

- Similarly if we are given acceleration as a function of time, we can integrate to find velocity:

$$\int a(t)\,dt = v(t) + C$$

Concept Check 3.2

A particle moving in a straight line starts from rest at the origin. Its acceleration $a\ ms^{-2}$ at any time t seconds is given by

$$a = 6 - 12t$$

(a) Show that the displacement function of the particle is given by **3**

$$x = \frac{1 + 6t + (1 - 2t)^3}{4}$$

(b) When and where does the particle change directions? **2**

4. Velocity As A Function Of Displacement

☐ The "Flip Trick" – Finding Displacement as a Function of Time

- When given $v = f(x)$ then $v = \frac{dx}{dt} = f(x)$. We cannot simply integrate this with respect to time since the velocity is presented as a function of x.
- Hence we use the "flip trick":

$$\frac{dx}{dt} = f(x) \text{ then } t = \int \frac{1}{f(x)} dx$$

- Use the given conditions to find the constant.

Example

When a particle is at position x cm, its velocity, v cm/s, is given by $v = (1 + 3x)$. Initially the particle is at the origin.

Find an expression for the position of the particle at any time t seconds. [6] 2

Step 1: Given $v = (1 + 3x)$, then $\frac{dx}{dt} = (1 + 3x)$

Hence $\frac{dt}{dx} = \frac{1}{(1+3x)}$, then integrate t with respect to x.

Step 2: Using the conditions $t = 0$, $x = 0$, find the constant, C

Concept Check 4.1

(a) The velocity of a particle in ms^{-1} at any position x, in metres, is given by $v = \sqrt{5x+1}$. Initially the particle is at the origin.

Note to students

Velocity is being expressed in terms of displacement x, not time t

(i) Find the displacement of the particle at any time t seconds. [7] 2

(ii) Hence find the velocity and acceleration of the particle at any time t seconds. [8] **2**

(iii) What is the initial velocity of the particle? [9] **1**

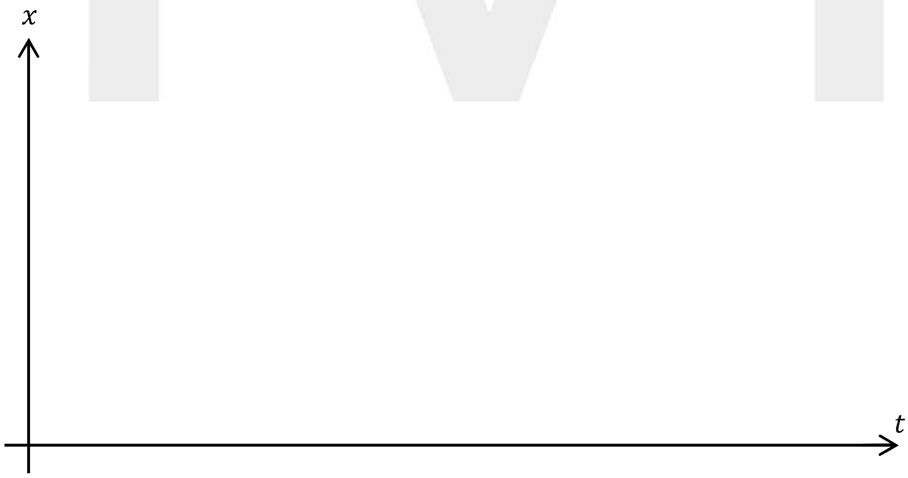
(iv) Hence describe the motion. [10] **1**

(b) A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O on the line and velocity given by $v = \sin x \cos x$. The particle starts $\frac{\pi}{4}$ metres to the right of O.

(i) Show that the displacement of the particle at any time t seconds is given by **2**

$$x = \tan^{-1}[e^t]$$

(ii) Find the limiting position of the particle and sketch the graph of x against t. [11] **3**



(c) The velocity, v cm/s, of a particle at position x cm, is given by $v = \frac{1+3x}{2}$.

(i) Find the acceleration of the particle when it is 2 cm from the origin. [12] **2**

(ii) Initially the particle is at the origin. Find an expression for the position of the particle at any time t seconds. [13] **3**

(iii) Hence find the initial velocity and acceleration of the particle. [14] **2**

(iv) Describe the motion of the particle. [15] **1**

(d) A falling ball experiences both gravitational acceleration g and air resistance that is proportional to its velocity. Thus a typical equation of motion is $\ddot{x} = -10 - 2v\ m/s^2$. Suppose that the ball is dropped from the origin.

Note to students

We will need to acquaint ourselves with this terminology. If an object is being 'dropped', what is the initial velocity?

(i) Find t as a function of v and hence find x as a function of t. [16] **3**

(ii) Describe the motion of the ball. [17] **1**