Year 12
Maths
Advanced

Lesson 6
Revisiting Exponential Growth and Decay

MATRIX
EDUCATION

1. Exponential Growth

☐ Definition

- Exponential **Growth** or **Decay** are terms that describe a special rate of change that occurs in many situations.
- **Exponential Growth** is the situation where the rate of increase of a quantity is directly proportional to the amount of the quantity present, that is:

$$\boldsymbol{\frac{dQ}{dt} = kQ}$$

- Where k is assumed constant and is called the **growth rate.**

If $k > 0$, then $\frac{dQ}{dt} > 0$ and Q is increasing.

If $k < 0$, then $\frac{dQ}{dt} < 0$ and Q is decreasing.

- This is also sometimes known as "Natural Growth" and "Natural Decay", as the population is changing by a ratio of the number of people in the population, which can be a good description of natural population dynamics.

☐ Formula for Exponential Growth

- Memorise the following formula, and proof of solution

$\boldsymbol{Q = Ae^{kt}}$ **is a solution to** $\boldsymbol{\frac{dQ}{dt} = kQ}$**.**

- To prove this, we use the following method: This is a common exam proof.

$$\frac{dQ}{dt} = kQ$$

$$LHS = \frac{dQ}{dt}$$
$$= \frac{d}{dt}(Ae^{kt})$$
$$=$$

$$RHS = kQ$$
$$=$$
$$= LHS$$

Example

The population of a town is increasing at a rate proportional to the existing population, where P is the population and t is time in years.

$$\frac{dP}{dt} = kP$$

(a) Show that $P = Ae^{kt}$ is a solution to the equation $\frac{dP}{dt} = kP$. 1

Using $\boldsymbol{LHS = RHS}$, show that $\boldsymbol{\frac{dP}{dt} = kP}$,where $\boldsymbol{P = Ae^{kt}}$.

(b) The initial population of the town is 8000. Ten years later, the population of the town is projected to be 15000. Find the exact values of A and k. [1] 2

Step 1: When $\boldsymbol{t = 0, P = 8\,000}$. Hence find $\boldsymbol{A}$.

Step 2: When $\boldsymbol{t = 10}$ and $\boldsymbol{P = 15000}$. Hence find $\boldsymbol{k}$.

(c) Determine the population in 20 years' time. [2] **2**

Using the answer for A and k from part (ii), find P when $t = 20$.

(d) When will the population be 30 000? [3] **1**

Solve for t when $P = 30\ 000$.

Discussion question

Suppose that population increases exponentially at 2% per year. Is k equal to 0.02? [4]

Concept Check 1.1

(a) The number of bacteria increases at a rate proportional to the number of bacteria present at any time t, measured in minutes i.e. $\frac{dN}{dt} = kN$.

Note to students

As soon as you see $\frac{dN}{dt} = kN$ you may use $N = Ae^{kt}$

(i) Show that $N = Ae^{kt}$ is a solution to the differential equation. **1**

(ii) The number of bacteria doubles in size every 19 minutes. Find the value of k. [5] **2**

Note to students

If you are missing the initial amount start with 100% .

(iii) How long will it take for the population to triple? [6] **2**

(b) At the beginning of 1985 a small African township reported 50 people suffering from a mysterious disease that was later identified as AIDS. By the end of 1987, the number of people suffering from the disease had increased to 120. Assume that the number of people suffering from AIDS, N, at any time t years is given by $\mathrm{N} = \mathrm{N}_0 \mathrm{e}^{\mathrm{kt}}$, where k is a constant.

(i) Calculate the values of N_0 and k. [7]

(ii) How many people in this African community were suffering from AIDS at the end of 2000? [8]

(iii) At the end of which year did the number of people in this community suffering from AIDS first reach 1000? [9]

(iv) At what rate was the number of people suffering from AIDS in this community increasing at the end of 2000? [10]