Year 11 Maths Extension 1

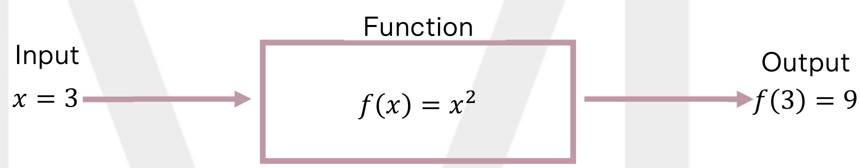
Lesson 8 Inverse Functions

MATRIX EDUCATION

2. Inverse Functions

☐ Introduction to Inverse Relations

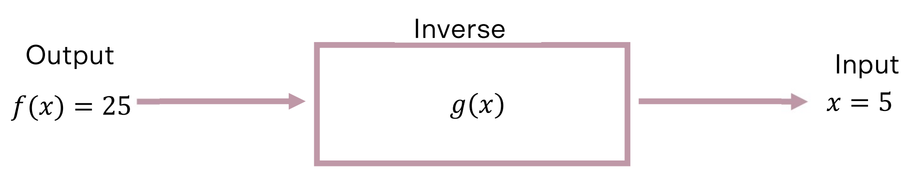
- Since relations are just sets of ordered pairs (x, y), the inverse relation is defined as the set of pairs where the **input, or independent variable x, is swapped with the output, or dependent variable y.**
  - For example, the inverse relation of $\{(1,3),(2,6)\}$ is $\{(3,1),(6,2)\}$.
- For relations that are defined by equations rather than sets, it helps to think about this process differently.
- We can imagine a function $f(x)$ to be a computer process, with the computer taking an input, x, and spitting out some number $f(x)$ based on a set of rules we've determined.



- A natural question that arises is:

> " If we have a certain output number – how do we find out what must have been input into our 'computer' ? "

- This is what we call the "inverse" of the function $f(x)$.
  - It is the process which returns an "input" for our specific "output".
  - The inverse of a function can be thought of as the **reverse process** of the original function, which **undoes** the function.



- So, in the example of $f(x) = x^2$ above, if we knew that the function had an output of 25, we could work out that $x = 5$ would have been an input.

Discussion

We saw that $x = 5$ would give an output of $f(x) = 25$. Is this the only x value which would give us 25 ? [1]

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- We can see that, though $f(x) = x^2$ has an inverse process (square rooting), we do not have unique solutions for each function output
  - Our two solutions correspond to $g(x) = \pm\sqrt{f(x)}$.
  - This would mean that, even though $f(x)$ has an **inverse relation**, this inverse would not be a **function**, as it would not pass the vertical line test.
  - This motivates the definition of an **inverse function,** $f^{-1}(x)$,

Example

For the relation $\{(0,1),(2,4),(7,4)\}$

(a) Is this relation a function? Why/why not? [2]

(b) Find the inverse relation. [3]

(c) Is the inverse relation a function? [4]

Solution

(a) ______________________________

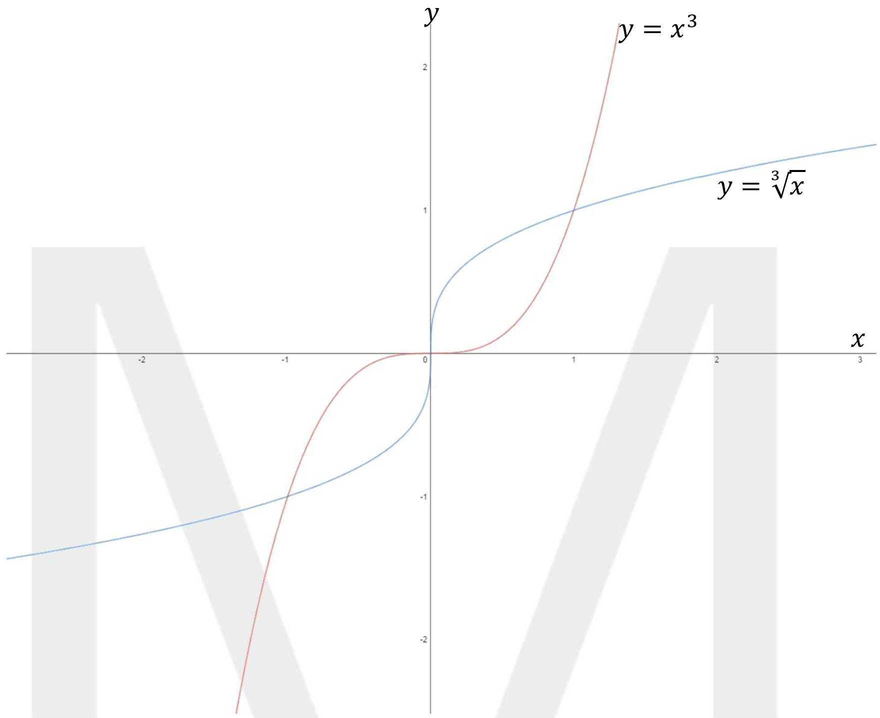
(b) ______________________________

(c) ______________________________

☐ Definition of Inverse Functions

- Recall that a function is defined as a rule changing an input value, x, to an output value, y, such that every x value has only one corresponding y value.

- Consider the pair of functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ as shown in the diagram below.

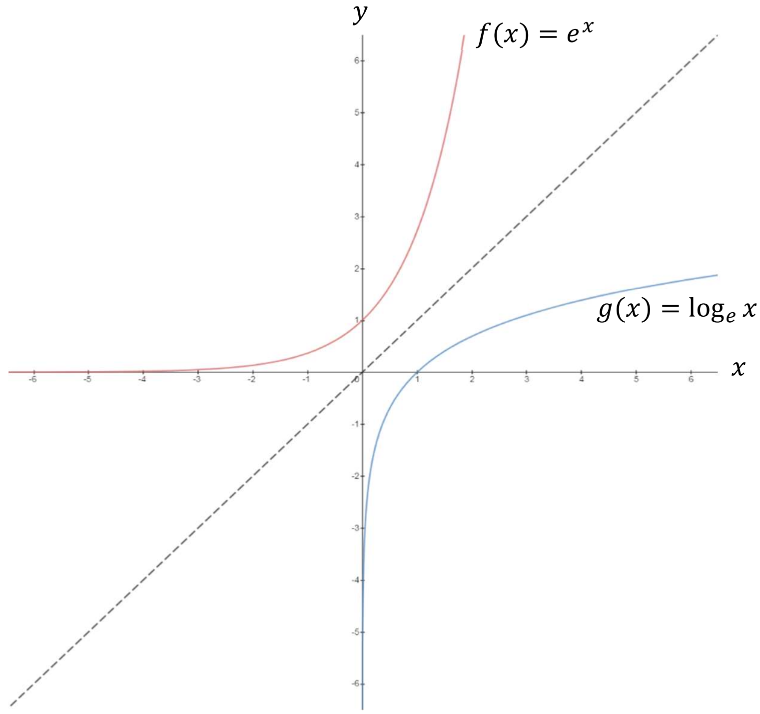


- Notice that:
  - $f(2) = 2^3 = 8$ and
  - $g(8) = \sqrt[3]{8} = 2$
- More generally
  - $f(x) = x^3$ and
  - $g(x^3) = x$
- Hence, we can see that f changes x to y and g changes y back to x.

> A pair of functions that behave in this way is called an inverse pair and $g(x)$ is called the **inverse function** of $f(x)$. $g(x)$ is written $f^{-1}(x)$.

- Alternatively, f and g are inverse functions if $f(g(x)) = g(f(x)) = x$.

- Using another example, consider the pair of functions $f(x) = e^x$ and $g(x) = \log_e x$ as shown in the diagram below.



- The logarithm function is often referred to as the **inverse** function of the exponential function.
- To find the inverse of a function, the pronumerals x and y are **interchanged** and the equation is **rearranged** to make y the subject. ‘
- Consider the exponential function $y = e^x$. The equation of its inverse function, the logarithmic function, can be derived in the following way.
  - Step 1: Interchange $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$y = e^x$$
$$x = e^y$$

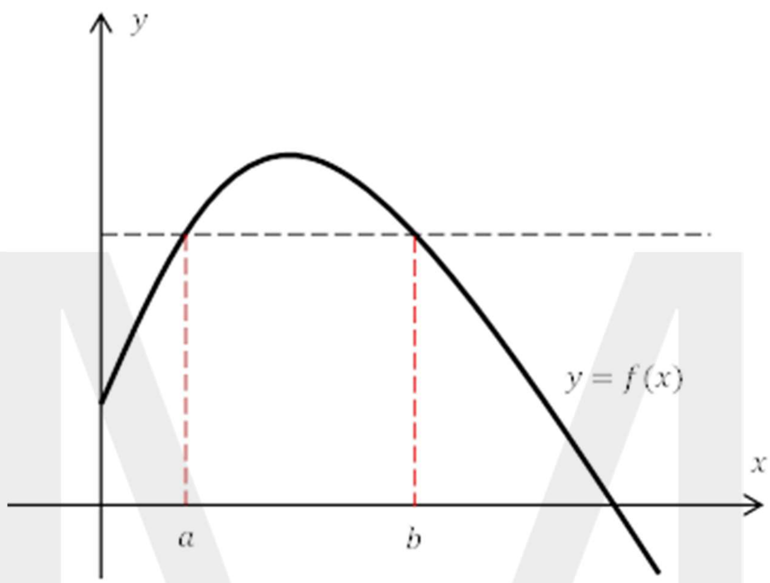
  - Step 2: Make $\boldsymbol{y}$ the subject:

$$e^y = x$$
$$y = \log_e x$$

- Throughout this lesson, this is a method we'll continue to explore in finding the inverse function of a given function.

☐ One to One Correspondence

- Question: When does a function have an inverse? What is the domain of our inverse function?
- A function has **one-to-one correspondence** if every y-value has only one x-value.



- The horizontal line in the diagram intersects the graph of $y = f(x)$ in two points. Hence there are two x-values for this y value.
- This is an example of a function that does not have one-to-one correspondence.

Discussion

What happens if we interchange the x and y-values of the function in the diagram above? Is the new curve a function? [5]

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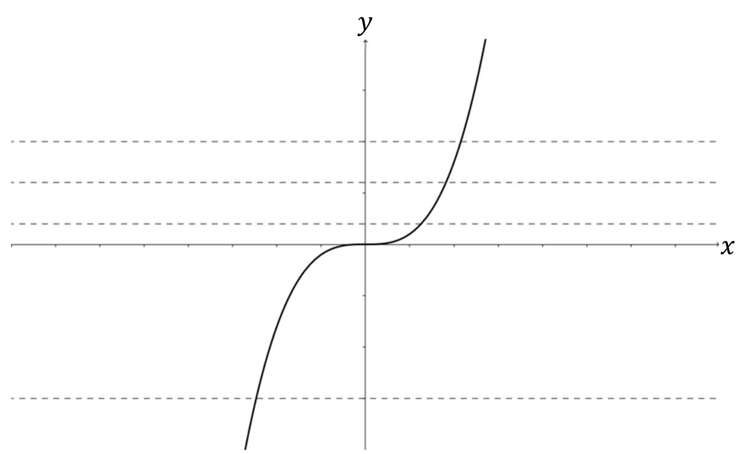
☐ Horizontal Line Test

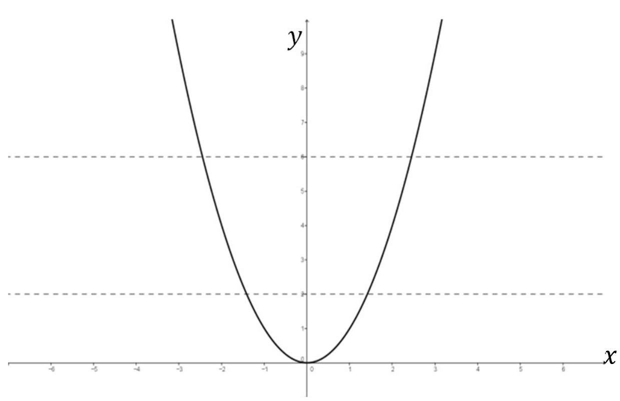
- If every horizontal line drawn to the curve $y = f(x)$ intersects the curve in at most one point, then the function $\boldsymbol{f(x)}$ is said to have one-to-one correspondence.

All strictly increasing or decreasing functions have one-to-one correspondence so these functions will have an inverse function.

- The curve $y = x^3$ is an increasing function. When the horizontal line test is applied, the line intersects the curve at only one point. Hence $f(x) = x^3$ has one to one correspondence and an inverse function exists.



- When the horizontal line test is applied to the parabola $y = x^2$, **there are two $\boldsymbol{x}$ −values for every $\boldsymbol{y}$ −value if $\boldsymbol{y > 0}$.** The function $f(x) = x^2$ does not have one-to-one correspondence so it **does not have an inverse function.** An inverse function will only exist if its domain is restricted to either $x \geq 0$ or $x \leq 0$ for this example.



Discussion

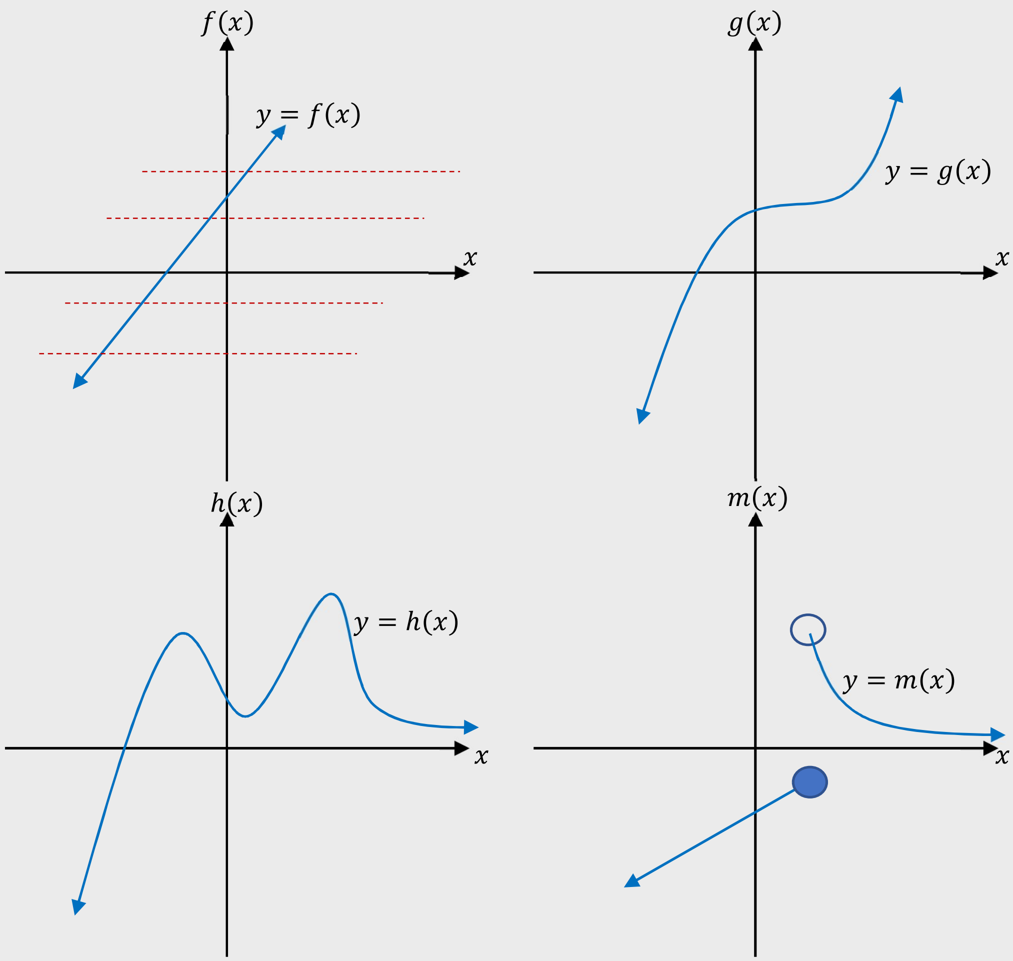
Can you give examples of other functions that have one-to-one correspondence ? [6]

Example

Using the horizontal line test, which of the following functions below

(a) Has one – to – one correspondence [7]

(b) Will have an inverse function $f^{-1}(x)$ [8]



Solution

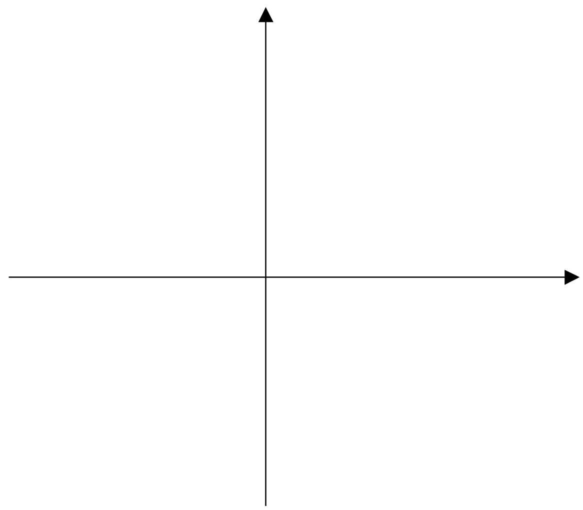
For $y = f(x)$, we see that a horizontal line only touches the graph once, and so has one – to – one correspondence . $\therefore$ $f(x)$ has an inverse function $f^{-1}(x)$

Complete the others yourself:

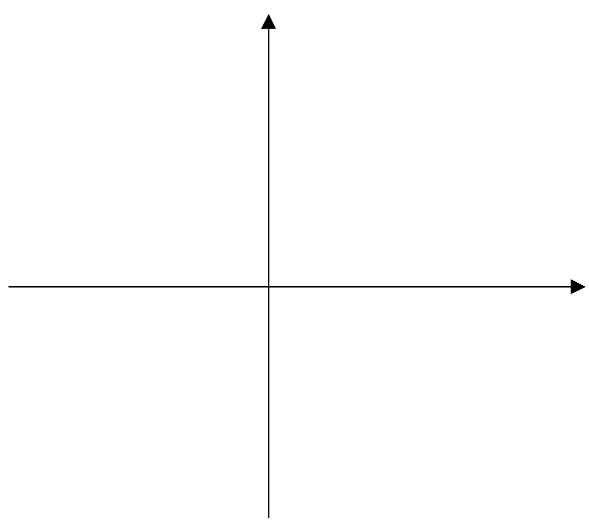
Concept Check 2.1

By sketching the graph of $y = f(x)$, determine whether the inverse function, $f^{-1}(x)$, exists.

(a) $f(x) = 2x - 1$ [9] 1



(b) $f(x) = 3x^2 - 2$ [10] 1



(c) $f(x) = -x^2 - 1$ for $x \geq 0$ [11] 1

> Note to students
>
> Do not ignore the domain restriction! It plays a very important role in the existence of $f^{-1}(x)$.