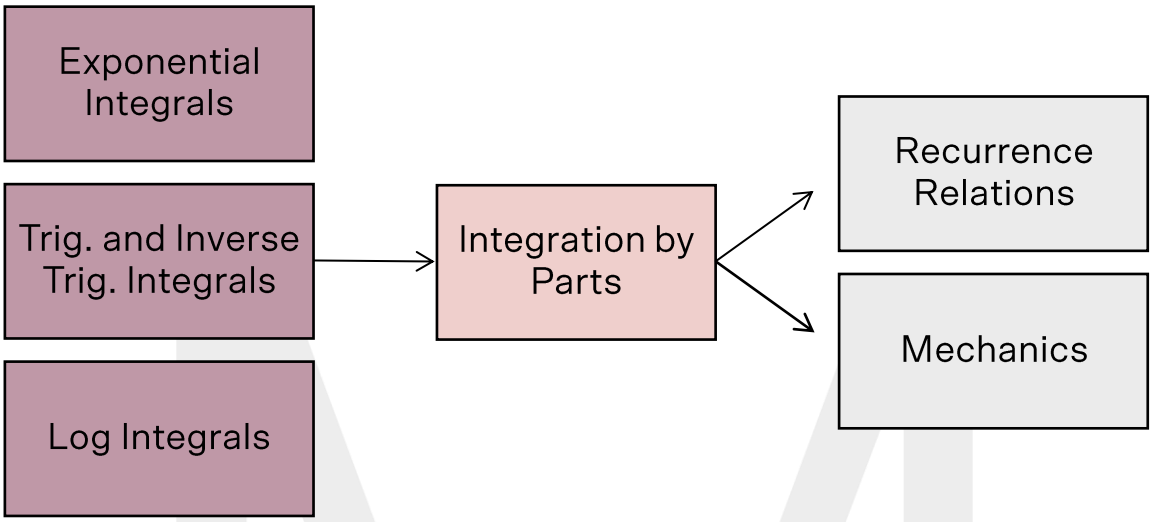
Year 12 Maths Extension 2

Lesson 1 Integration by Parts and Introduction to Reduction Formula

MATRIX
EDUCATION

1. Overview Of Methods Of Integration 4

☐ Introduction to Integration by Parts



$$\int udv = uv - \int vdu$$

- This lesson will examine the theory of integration by parts and its use in reduction formulae. Integration by parts is the closest that we come to having a product rule for integration. But unlike the product rule for differentiation, which always works, integration by parts only works at specific situations. The typical parts problem has the format: $\int ("little\ poly") \times ("nice\ function")dx.$

- For example

$$\int (4x)(e^{2x})\,dx \quad \text{or} \quad \int_0^{\pi} x^2 \sin(x)\,dx$$

- The "nice" function is usually a log, trig. or exponential function. The parts technique is however quite flexible and it can also be used in some other quite surprising situations. An important thing to keep in mind with parts is that it doesn't solve the problem! The process of integration by parts (like substitution) converts an integral into a secondary problem which hopefully can then be done. Remember also that not every product can be integrated by parts.

- We have actually seen integration by parts before. Remember those "differentiate, hence integrate" questions from Maths Extension 1 and Maths Advanced? Those are sometimes actually getting you to use the integration by parts formula to evaluate unknown integrals.
- Even though integration by parts can help us evaluate hard integrals, it can by no means calculate every integral, functions like $\sin x^2$ and e^{x^2} famously have no "nice" (or explicit) primitive.
- In fact, if we incorrectly choose which term should be u and which should be v, it can actually make the integral harder to evaluate, as this comic from xkcd puts:

- Source: xkcd.com; http://imgs.xkcd.com/comics/integration_by_parts.png
- We will then introduce reduction formulae (also called recurrence relations) and will examine these in more detail next lesson.
- Reduction formulae are like little integration machines which can be used recursively to evaluate very complicated integrals. These questions are usually presented in two parts. Part i) will ask you to prove a recurrence formula while part ii) will then ask you to use the recurrence relation on a particular integral.

2. Integration By Parts

☐ Introduction to Integration by Parts

- To differentiate the product of two functions, we use the product rule.
- To integrate the product of two functions, we can use integration by parts.

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

☐ Proof:

- Consider the product rule of differentiation.

$$\frac{d}{dx}(uv) = u'v + v'u \qquad \leftrightarrow \qquad \frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

- Make $u\,dv$ the subject.

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

- Integrate both sides with respect to x.

$$\int u\,dv = uv - \int v\,du$$

Note to students

One of the signs of the problem potentially being an integration by parts problem is if your integral is the product of two different 'types' of functions. Below is an example of an integral requiring integration by parts.

$$\int x \ln x\,dx$$

Before attempting this method, always check that the integral can be found by the use of an appropriate substitution. The below integral does not require the use of Integration by Parts.

$$\int x\cos(x^2)\,dx$$

Example 1

Evaluate [1]

$$\int xe^x\,dx$$

Solution:

Step 1: Establish what is u and dv.

We can either choose:

$\boldsymbol{u = x}$	$\boldsymbol{dv = e^x\,dx}$
$\boldsymbol{du = 1\,dx}$	$\boldsymbol{v = e^x}$

Giving us $\int xe^x\,dx =$ _____

OR

$\boldsymbol{u = e^x}$	$\boldsymbol{dv = x\,dx}$
$\boldsymbol{du = e^x\,dx}$	$\boldsymbol{v = \frac{x^2}{2}}$

Giving us $\int xe^x\,dx =$ _____

Pick the one that yields a more friendly integral, and evaluate the Integral.

_____

_____

_____

Did you know?

You've actually SEEN integration by parts before!

Some "differentiate, hence integrate" question are actually integration by parts questions in disguise!

Example 2

Evaluate [2]

$$\int x \ln x \, dx$$

Solution:

Step 1: Establish what is u and dv.

We don't even know what $\int \ln x \, dx$ is! So choosing $dv = \ln x$ is obviously a bad idea in that respect.

Step 2: Use the formula for Integration by Parts to simplify the integral.

Step 3: Evaluate the remaining integral to acquire the final result.

Concept Check 2.1

(a) $\int 3xe^{4x}\,dx$ [3] 3

(b) $\int 2x\cos 5x\;\,dx$ [4] 3