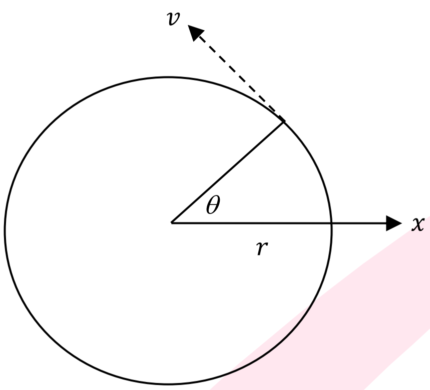
# 2. UNIFORM CIRCULAR MOTION

## ❐ Definition of Angular Velocity

- When a particle moves in a circle and sweeps an angle, $\theta$ radians, at the centre of the circle in time $t$ seconds, the rate of change of $\theta$ with respect to time is called its **angular velocity**, and is usually denoted by $\omega$.




**Angular Velocity** $= \frac{d\theta}{dt} = \dot{\theta} = \omega$

- Units for angular velocity are radians/second.

**Did you know:**

1 Revolution per second = $2\pi$ radians per second.

## ❒ Tangential Velocity (Velocity Along the Arc)

- The linear velocity, $v$, of the particle $P$, is a vector in the direction of the motion and is directed along the tangent at $P$.
- We know from the 2 unit course that the arc length is given by $s = r\theta$. Differentiate both sides with respect to $t$.

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta)$$

$$v = r \times \frac{d\theta}{dt}$$

- This gives us the very important relationship that:

| **Relationship between linear and angular velocity** |
|---|
| $\boldsymbol{v = r\omega}$ |

**Note to Students:**
The magnitude of the linear velocity $v$ can be thought of as the reading on your speedometer if you were driving around the circle.

**Note to Students:**
You absolutely <u>must</u> remember the relationship that $v = r\omega$. It will be used often in this course.

- The **linear velocity** gives the **s**............. [1] at which $P$ moves along the circumference whereas the **angular velocity** gives the number of **r**......................... [2] **per second** through which $P$ travels.

## ❐ Constant Angular Velocity

- If the particle moves around the circle at a constant speed, then the particle is said to be moving in **uniform circular motion**, and $\omega$ is constant.

| **Constant Angular Velocity** |
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| $$\frac{d\omega}{dt} = 0$$ $$\therefore \theta_{\text{travelled}} = \omega t$$ |

- The **period** of a circular motion with constant angular velocity $\omega$ is given by $T = \frac{2\pi}{\omega}$, and is measured in seconds, minutes, hours, etc**.**

| **Period of Uniform Circular Motion** |
| --- |
| $$T = \frac{2\pi}{\omega}$$ |

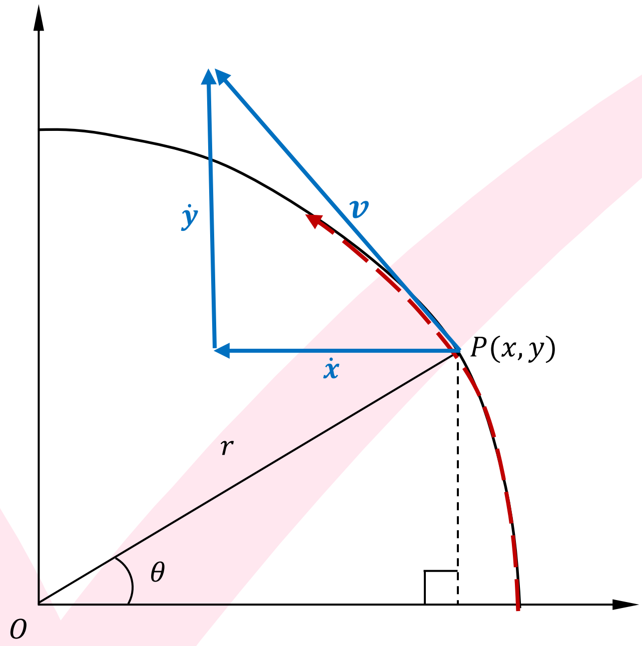
**Did you know:**
In this course we will usually deal with constant angular velocity, with $\dot{\omega} = 0$.

## ❐ Horizontal & Vertical Components of Velocity & Acceleration

- In this section we will derive the two formulas for centripetal force.

- The diagram below shows a particle moving in uniform circular motion.

- The tangential velocity vector, $v$, and its horizontal and vertical components $\dot{x}$ and $\dot{y}$ are shown in blue.




- In the diagram, $x = r\cos\theta$ and $y = r\sin\theta$. The gradient of $OP$ is $\tan\theta$.

- Using the chain rule to differentiate $x$ and $y$, the horizontal and vertical components of velocity are respectively given by:

$$\dot{x} = -r\sin\theta \times \frac{d\theta}{dt} = -r\omega\sin\theta$$

$$\dot{y} = r\cos\theta \times \frac{d\theta}{dt} = r\omega\cos\theta$$

  – Similarly to projectile motion, the magnitude $v$ and angle $\alpha$ of the linear velocity, with respect to $\dot{x}$ and $\dot{y}$, are respectively given by

$$v^2 = \sqrt{\dot{x}^2 + \dot{y}^2} \qquad \tan\alpha = \frac{\dot{y}}{\dot{x}}$$

  – And hence we find an alternative proof to the formula covered earlier

$$\begin{aligned} v^2 &= \sqrt{r^2\omega^2\sin^2\theta + r^2\omega^2\cos^2\theta} \\ &= \sqrt{r^2\omega^2} \\ \Rightarrow v &= r\omega \end{aligned}$$

  – To find the gradient of the velocity vector

$$\begin{aligned} \tan\alpha &= \frac{\dot{y}}{\dot{x}} \\ &= -\cot\theta \end{aligned}$$

  – And note that this is perpendicular to $OP$ since the gradient of $OP$ is $\tan\theta$ and the product of their gradients is $-1$, so the velocity vector points along the tangent.

- We note that $\dot{x}$ and $\dot{y}$ are defined in terms of the variable $\theta$, and so the chain rule, and product rule, must be used to differentiate in terms of time, to fine $\ddot{x}$ and $\ddot{y}$.

- The horizontal and vertical components of <u>acceleration</u> are respectively:

$$\ddot{x} = \frac{d}{dt}(-r\omega \sin\theta) = -r\frac{d}{dt}(\omega \sin\theta) = -r\left(\dot{\omega}\sin\theta + \omega\cos\theta \times \frac{d\theta}{dt}\right)$$

$$= -r(\dot{\omega}\sin\theta + \omega^2\cos\theta)$$

  – But, as we are dealing with constant angular velocity, $\dot{\omega} = 0$, and so:

$$\dot{x} = -r\omega^2\cos\theta$$

  – Similarly for the vertical component:

$\ddot{y} =$ .......................................................................... [3]

  – Similarly to how we calculated the magnitude of the linear velocity, the magnitude $a$ of the linear acceleration is found using pythagorus.

$a =$ ......................................................................

$=$ ...................................................................... [4]

- Similarly to how we calculated the direction of the linear velocity, the angle $\beta$ of the linear acceleration is found using

$$\tan\beta = \frac{\ddot{y}}{\ddot{x}}$$

$=$ .................................

$= \tan\theta$

- Hence $\beta =$ ........................ [5], which is pointing to the centre.

- This acceleration forms the basis of a force called **centripetal force**

## ❒ Centripetal Force

**Centripetal Force**

$$F_c = mr\omega^2$$

$$= \frac{mv^2}{r} \text{ using } v = r\omega$$

And it is directed towards the centre of motion.

- This gif http://www.regentsprep.org/regents/physics/phys06/bcentrif/lrlmlv.gif demonstrates the centripetal force on a ball as it is swung round in a circle.

**Did you know:**

The name 'centripetal' comes from the latin "*centrum petere"* – centre seeking.

This is instead of the, incorrect, fictitious "centrifugal" force – centre fleeing.

**Example**

A 2 kg ball is swung round in a circle on a 3 metre rope. The angular velocity of the ball is 50 revolutions per minute. Find the:

(a) Angular velocity of the ball in radians/sec. **1**

(b) Tangential speed of the ball. **1**

(c) Tension in the rope. **1**

## ❒ Summary of Facts and Formulas for Uniform Circular Motion

**Angular Velocity**

Angular velocity $= \omega$

Period $T = \frac{2\pi}{\omega}$

$\omega$ is assumed to be constant

**Linear Velocity**

$v = r\omega$

Acts along the tangent

**Acceleration**

$$a = r\omega^2 = \frac{v^2}{r}$$

Acts towards the centre of the circle.

**Centripetal Force**

$$F = mr\omega^2 = \frac{mv^2}{r}$$

**Note to Students:**

You will use both forms of centripetal force, depending on the problem at hand.

We usually use $mr\omega^2$ in *circular motion on a flat surface* and *conical pendulums* for our centripetal force.

However, in *banked tracks*, we usually use $\frac{mv^2}{r}$ for our centripetal force.

## Concept Check 2.1

(a) A record has a radius of 15 cm and it rotates at an angular velocity of 33 revs/min.

Find the:

(i) Angular velocity in radians/sec. [6] **1**

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(ii) Linear speed of a point $P$ on the rim of the record. [7] **1**

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(iii) Acceleration of the point $P$. [8] **1**

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(b) A flywheel of radius 60 cm makes 300 revolutions per minute. Find the:

(i) Linear velocity (in terms of $\pi$) of a point $P$ on the rim. [9] **1**

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(ii) Acceleration of the point $P$ in terms of $\pi$. [10] **1**

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(c) Assuming that the earth is a sphere of radius 6400 km and it rotates about its axis once every 24 hours, find the acceleration of any given point on the equator. [11] **2**

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(d) The second hand of a watch is 2 cm long and rotates continuously.

Find the:

(i) Angular velocity of the tip of the second hand. [12] **1**

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(ii) Linear velocity of the tip of the second hand. [13] **1**

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(iii) Acceleration of the tip of the second hand. [14] **1**

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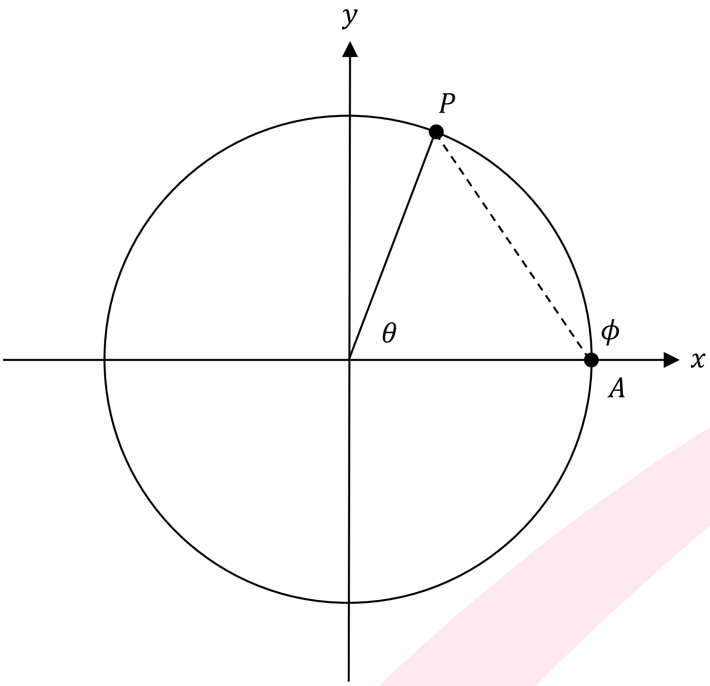
**Discussion Question:**

What if we measured everything from say the middle of the second hand instead of the tip? Will our answers be the same as above? [15]

(e) Consider a point $P$ on a circle centred at $O$ and positive $x$ intercept $A$. Let $\angle POA = \theta$ and let $\phi$ be the angle that $AP$ makes with from the positive $x$ axis.




The point $P$ moves anticlockwise along the circumference of the circle with uniform angular velocity $\omega$.

Show that $= 2 \times \frac{d\phi}{dt}$ . **1**

(f) A satellite travels in a circular orbit of radius $R$ km. It takes $T$ minutes to complete an orbit. Find the:

(i) Angular velocity in radians per hour. [16] **1**

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(ii) Tangential velocity in kilometres per hour. [17] **1**

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