2. PERMUTATIONS

❒ Multiplication Principle

- The topic of permutations develops efficient techniques to determine the number of different ways certain events can happen.
  - The events will occur in succession and the order in which they occur matters.
- If the first event can happen in n_1 ways, a second event can happen in n_2 ways, a third event in n_3 ways and so on, then the number of ways for these events to occur in sucession is:

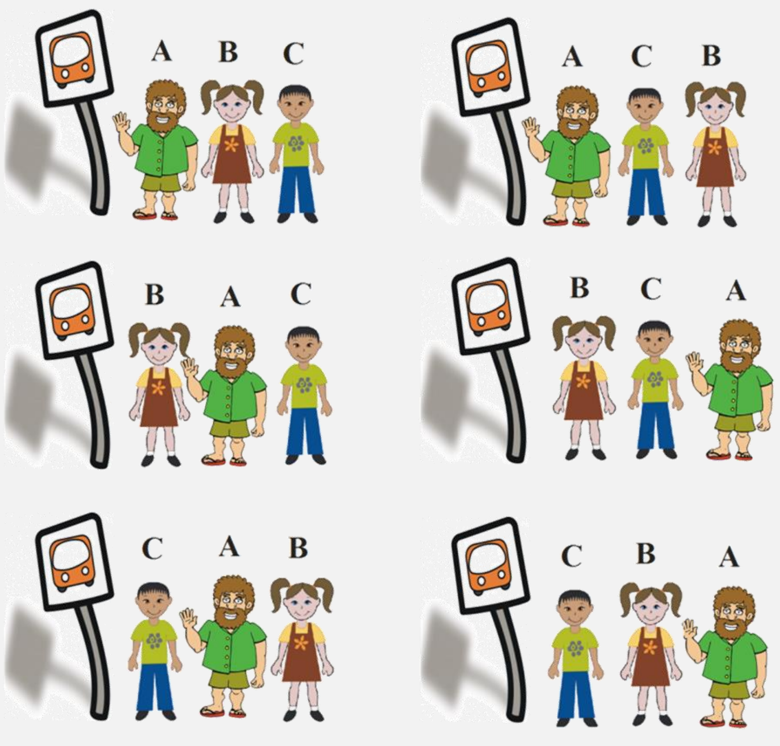
$$n_1 \times n_2 \times n_{3\times\ldots}\ldots.$$

Example 1

Suppose that three friends, Adam (A), Betty (B) and Charles (C), line up for a bus. In how many ways can this be done?

METHOD 1: LISTING

Since the number of people is quite small, we can simply list the possibilities.



Clearly there are only 6 choices: ABC, ACB, BAC, BCA, CAB and CBA. We say that A, B and C can be **permuted in 6 different ways**.

As you can imagine, actually listing the permutations is difficult if we have more people lining up! A better method is to instead consider the options for each successive position.

METHOD 2: MULTIPLICATION PRINCIPLE

Step 1: Set up a template.

Step 2: We have 3 options for the first position, A, B or C.

3		

Step 3: We now only have 2 options left for the second position. Note that the same person we chose for the first position cannot be chosen again.

3	2	

Step 4: There is only one choice left for the last position.

3	2	1

Step 5: Multiply the outcomes.

$$3 \times 2 \times 1 = 6$$

This method can now be used to answer much more difficult questions!

Example 2

In how many ways can 8 people line up for a bus?

Solution

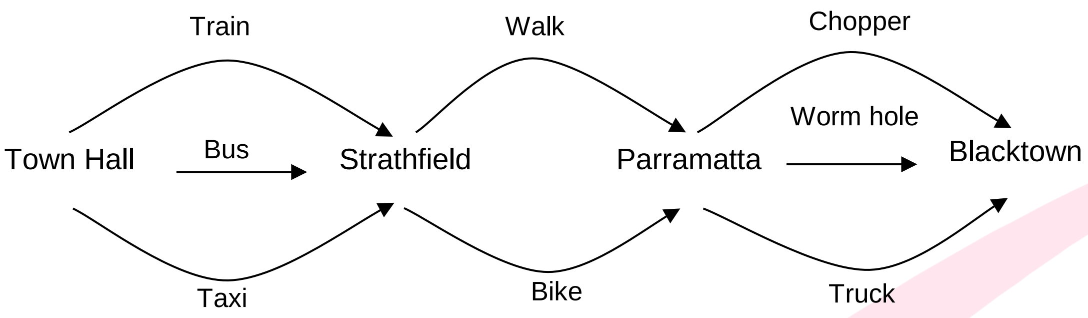
Use the multiplication principle.

8	7	6	5	4	3	2	1

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$$

Concept Check 2.1

(a) A student wishes to travel from Town Hall to Blacktown via Strathfield and Parramatta. The number of ways of travelling from Town Hall to Blacktown are shown in the diagram below. How many ways are there of travelling from Town Hall to Blacktown? [1] **1**



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(b) A café's menu had 2 soups, 4 entrees, 3 main courses and 4 desserts. Chris was very hungry and decided to choose a four course lunch. How many different lunches are available to Chris if he orders:

(i) A dish from each course? [2] **1**

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....

(ii) An entrée as a main course? (assume Chris can eat the same dish twice) [3] **1**

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(c) Ada's wardrobe consists of 6 pairs of jeans, 8 blouses and 3 sports jackets. How many different arrangements of clothing are available to Ada each day, assuming that she always wears a pair of jeans, a blouse and a sports jacket? [4] **2**

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❐ Ordered Arrangements With Repetition

- In counting ordered selections with repetition allowed, the number of objects to choose from does **not** change as each event is completed.
- Using the multiplication principle, we can imagine each of the n objects being able to be placed into each of the k boxes, and so the number of selections is:

Ordered arrangements of n objects into k boxes with repetition is n^k

Example

How many different:

(i) 2-digit numbers [5]

(ii) 3-digit numbers [6]

(iii) 4-digit numbers [7]

are possible from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if **repetition is** allowed?

Solution

(i)

First digit	Second digit

....

(ii)

First digit	Second digit	Third digit

....

(iii)

First digit	Second digit	Third digit	Fourth digit

....

NOTE TO STUDENTS

- Use boxes for counting ordered selections.
- Remember that the boxes are in the order in which the selections are made.

❐ Ordered Arrangements Without Repetition

- In counting ordered selections without repetition, the number of objects to choose from decreases by 1 as each event is completed.
- This means that the number of arrangements of n objects into k boxes will be $n \times (n-1) \times (n-2) \times \ldots . \times (n-k+1)$. This is sometimes written as

Ordered arrangements of n objects into k boxes without repetition
$n(n-1)(n-2)\ldots(n-k+1) = {}^nP_k$

- Where ${}^nP_k$ stands for "n permute k".

DID YOU KNOW?

It is the examiner's job to specify whether or not repetition is allowed.

Example

How many different:

(i) 2-digit numbers [8]

(ii) 3-digit numbers [9]

(iii) 4-digit numbers [10]

are possible from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if **repetition is NOT** allowed?

Solution

(i)

First digit	Second digit

....

(ii)

First digit	Second digit	Third digit

....

(iii)

First digit	Second digit	Third digit	Fourth digit

....

Concept Check 2.2

(a) A car number plate has three letters of the alphabet followed by three digits selected from the digits $1, 2, \ldots, 9$. How many different number plates are possible if repetition of the letters and digits is not allowed? [11] **1**

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(b) Using the letters of the word $SECTION$, how many:

(i) Three-letter codes. [12] **1**

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....

(ii) Four-letter codes. [13] **1**

....

....

(iii) Six-letter codes can be formed without repeating any letter? [14] **1**

NOTE TO STUDENTS
Always check for repetition. There is none here.

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(c) Four flags are used to send messages from a vertical flagpole. Find the total number of possible messages that can be sent:

(i) If all 4 flags must be used. [15] **1**

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(ii) If at least 1 flag must be used. [16] **2**

NOTE TO STUDENTS
As a general rule we use $\times$ for "and" and $+$ for "or".

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3. ORDERED ARRANGEMENTS OF n ELEMENTS

❐ Ordered Arrangements Without Restrictions

- Assume there are n different elements. These n elements are to be arranged in order in a line. To do this, an element is selected one at a time at random (without replacement) from the set:

No. of the selection	1^{st} selection	2^{nd} selection	……….	$(n-2)^{th}$ selection	$(n-1)^{th}$ selection	n^{th} selection
No. of ways						

  – $\mathbf{1}^{st}$ **Selection**: It can be chosen at random from any of $\boldsymbol{n}$ **different elements**. Therefore, there are ……….. ways in which the first space can be occupied.
  – $\mathbf{2}^{nd}$ **Selection**: Since one of the elements has been selected, the second element can be chosen from any of the remaining $(\boldsymbol{n}-\mathbf{1})$ **different elements**. Therefore, there are ……….. ways in which the second space can be occupied.
  – $\mathbf{3}^{rd}$ Selection:

  ……………………………………………………………………………………………

  ……………………………………………………………………………………………

  – $(\boldsymbol{n}-\mathbf{1})^{th}$ **Selection:** Since $(n-2)$ elements have been used, there are only 2 elements left. Therefore, there are ………. ways in which $(n-1)^{th}$ space can be occupied
  – $\boldsymbol{n}^{th}$ **Selection:** Since $(n-1)$ elements have been used, there is only 1 element left. Therefore, there is ………. way in which n^{th} space can be occupied

- Hence the number of ordered selections (without replacement) of n elements from a set of n elements is $\boldsymbol{n} \times (\boldsymbol{n}-\mathbf{1}) \times (\boldsymbol{n}-\mathbf{2}) \ldots\ldots \mathbf{3} \times \mathbf{2} \times \mathbf{1} = \boldsymbol{n}!$
- This is called a **PERMUTATION** of n elements selected from a set of n different elements.

Number of Ordered Arrangements of n Objects Without Restriction $= n!$

Example

How many three-letter "words" can be arranged from the letters A, B and C without repetition? [17]

Solution

Method 1: Writing all the possible outcomes.

ABC	ACB				

Therefore, there are ....

Method 2: Using the factorial notation.

The number of ordered selections of n elements from a set of n elements is

$$n \times (n-1) \times (n-2) \ldots\ldots 3 \times 2 \times 1 = n!$$

Therefore, the number of ordered selections of 3 elements from a set of 3 elements is

$$3 \times 2 \times 1 = 3! = 6$$

Concept Check 3.1

(a) How many four-letter codes can be formed from the letters $A, B, C,$ and D without repetition? Use the factorial notation method. [18] **1**

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....

(b) How many arrangements of a standard deck of cards are there? Use the factorial notation method. [19] **1**

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❐ Ordered Arrangements With Restrictions

- These are the most common type of permutation and combination questions, where there is a restriction on how we can arrange our set.
- When a problem presents to you any restrictions, always deal with any restrictions first, then arrange the remaining elements.

Example

5 boys and 4 girls are lining up at the school canteen. How many arrangements can be made if:

(i) The boys and girls are to alternate? [20]

(ii) Three boys must be in the front of the line? [21]

Solution

(i)

Since the boys and girls must alternate, the only way that this is possible is for a boy to be at the front of the line.

The first position can be filled by any of the 5 boys. The next position can be filled by any of the 4 girls. The second position for the boys can be filled by any of the remaining 4 boys. The second position for the girls can be filled by any of the remaining 3 girls, etc. Complete the following.

B	G	B	G	B	G	B	G	B
5	4							

Hence there are ……………………………………………………….......... arrangements.

(ii)

Since three boys must be at the front, then fill these positions first.

First position can be filled by any of the 5 boys, then the next position by any of the remaining 4 boys and the third position by any of the remaining …………. boys.

From the fourth position onwards, there are no restrictions. Hence the fourth position can be filled by any of the remaining 6 people. Complete the table

B	B	B						
5	4							

Hence there are ……………………………………………………….......... arrangements.

Concept Check 3.2

5 boys and 4 girls are lining up at the school canteen. How many arrangements can be made if:

(i) The line begins and ends with a boy? [22] **2**

(ii) The girls are together? [23] **2**

NOTE TO STUDENTS

In some permutation problems, particular members must be grouped together. In such cases you must use the following rules.

- Arrange the groups,
- Arrange the individuals within each group. Note: A group may consist of a single individual.

In our question:

If the girls need to be positioned together, "glue" them together into a 'big girl'. You then have 6 objects to permute: $b_1, b_2, b_3, b_4, b_5, G$

(iii) All the boys are together and all the girls are together? [24] **2**

B	B	B	B	B	G	G	G	G
5	4							

OR

G	G	G	G	B	B	B	B	B
4	3							

- There are two groups that can be arranged in ………… ways.
- The number of ordered arrangements for the boys in a group is ………… and the number of ordered arrangements for the girls in a group is …………..
- **Therefore the required number of ordered arrangements is** ………………………….

(iv) Two particular girls X and Y wish to be together? [25] **2**

NOTE TO STUDENTS

"Glue" the two girls together.

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