4. **THE BINOMIAL COEFFICIENT \( \binom{n}{k} \)**

**Expansion using the Binomial Coefficient**

- In the previous section, we saw that

\[
(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n
\]

- Where the coefficients can be found using Pascal’s Triangle.

- When \( n \) is a large integer, the use of Pascal’s Triangle becomes tedious and we need a more efficient approach

  - The first step is to establish a notation for the coefficients in the above expansion. The \( n + 1 \) numbers are called **Binomial Coefficients** and are denoted (in order):

\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}
\]

So \((x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\)

**Example**

Write down the expansion of \((x + y)^5\) using binomial coefficients.

**Solution**

**Step 1:** Write out the powers of \( x \) and \( y \) first

\((x + y)^5 = \quad x^5 + \quad x^4y + \quad x^3y^2 + \quad x^2y^3 + \quad xy^4 + \quad y^5\)

**Step 2:** Insert the binomial coefficients in the form \( \binom{n}{k} \)

\[\therefore (x + y)^5 = \quad \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5\]
Other Binomial Expansion Notation

- Other than \( \binom{n}{k} \) there is another notation used in the HSC for binomial expansions: \( nC_k \).

**Note to Students:**
This is the same “n choose k” notation used in your permutations and combinations course learnt last term.

- You must be familiar with both, as they are both used in exams.

\[
\binom{n}{k} \equiv nC_k
\]

**Example**

Write the expansion of \((x + y)^5\) in \( nC_k \) notation \(^7\)

**Solution:**

\[(x + y)^5 = \ldots\]

**Concept Check 4.1**

(a) Expand fully \((5 + x)^5\) showing all the coefficients in the form \( \binom{n}{k} \) \(^8\) 2
(b) Consider the binomial expression \((4 - 3x)^{10}\)

(i) Expand \((4 - 3x)^{10}\) as far as the 5th term using the notation \(\binom{n}{k}\). [9] 

(ii) Deduce the 8th term using the notation \(\binom{n}{k}\) from (i). [10] 

**Note to Students:**

The 8th term is when \(k = 7\), as the first term in the expansion is \(k = 0\).
(c) Show by expansion and using the notation \( ^nC_k \):

\[ (2 + 3x)^4 = \sum_{k=0}^{4} 4C_k (2)^{4-k} (3x)^k \]

**Note to Students:**

We have two distinct notations for the Binomial coefficients \( ^nC_k \) and \( \binom{n}{k} \). You must be familiar with both, as they are both used in the HSC.
5. **A FORMULA FOR** \( \binom{n}{k} \)

☐ **Review of Factorial Notation**

- **Definition of factorial notation.**
  
  \[ n! = n(n - 1)(n - 2)(n - 3) \ldots 3 \times 2 \times 1 \]

- **0!**
  - Zero factorial is defined as 1
  - This can be obtained from Pascal’s triangle, where \( \binom{n}{0} = 1 \)

  \[ 0! = 1 \]

- **\((n + 1)n!\)**
  - \((n + 1)n! = (n + 1)[n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1]\)
  - Therefore

  \[ (n + 1)n! = (n + 1)! \]

- **n\((n - 1)!\)**
  - Just like \((n + 1)n! = (n + 1)!\), we have \(n(n - 1)! = n!\)
  - Therefore:

  \[ n(n - 1)! = n! \]

- **\(\frac{n!}{(n-1)!}\)**
  - \(\frac{n!}{(n-1)!} = \frac{[n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1]}{[(n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1]}\)
  - Therefore

  \[ \frac{n!}{(n-1)!} = n \]

- **\(\frac{n!}{(n-k)!}\)**
  - \(\frac{n!}{(n-k)!} = \frac{[n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1]}{[(n-k) \times (n-k-1) \times (n-k-2) \times \ldots \times 3 \times 2 \times 1]}\)
  - Therefore

  \[ \frac{n!}{(n-k)!} = n(n - 1)(n - 2) \ldots (n - k + 1) \]
Concept Check 5.1

(a) Write down the meaning of \(10!\) \([11]\)

(b) Simplify \(13 \times 12!\) \([12]\)

(c) Simplify \(\frac{15!}{12!}\) \([13]\)

(d) Express \(\frac{n!}{(n-5)!}\) without factorial notation. \([14]\)

(e) What is the value of \(\frac{15!}{9!}\)? \([15]\)

Discussion Question 1:
Find 80! Using your calculator. What happened?\([16]\)

Discussion Question 2:
What is the value of 0! ?\([17]\)
Investigation of $\binom{n}{k}$

- Consider the expansion

\[ (1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{k} x^k + \cdots + \binom{n}{n} x^n \]

- Differentiate both sides of the equation

- Let $x = 0$ on both sides of the equation in (i)

\[ \therefore \binom{n}{1} = \] 

- Differentiate both sides of the equation in (i) and let $x = 0$

\[ \therefore \binom{n}{2} = \] 

- Differentiate both sides of the equation in (iii) and hence find an expression for $\binom{n}{3}$

\[ \therefore \binom{n}{3} = \] 

- Observing the pattern above, what would you say is an equivalent expression for $\binom{n}{4}$ and $\binom{n}{5}$?