YEAR 12
MATHS EXT 1

LESSON 5: INVERSE TRIGONOMETRIC
FUNCTIONS 1

(Y12 Mathematics Ext 1 HSC Topic)
1. **MULTIPLE CHOICE QUESTIONS**

1. A function \( f(x) \) is given by the equation \( f(x) = 2 \sin^{-1}(3x) \).
   The domain and the range of \( f(x) \) are respectively: \([1]\)
   
   (a) \(-3 \leq x \leq 3\) and \(-2 \leq f(x) \leq 2\)
   (b) \(-\frac{1}{3} \leq x \leq \frac{1}{3}\) and \(-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}\)
   (c) \(-\frac{1}{3} \leq x \leq \frac{1}{3}\) and \(-\frac{\pi}{4} \leq f(x) \leq \frac{\pi}{4}\)
   (d) \(-\frac{1}{3} \leq x \leq \frac{1}{3}\) and \(-\pi \leq f(x) \leq \pi\)

2. The domain for \( 3 \cos^{-1} 4x \) is \([2]\)
   
   (a) \(-\frac{3}{4} \leq x \leq \frac{3}{4}\)
   (b) \(-\frac{1}{3} \leq x \leq \frac{1}{3}\)
   (c) \(-\frac{1}{4} \leq x \leq \frac{1}{4}\)
   (d) \(-1 \leq x \leq 1\)

3. The domain and range of the function \( f(x) \), where \( f(x) = 3 \sin^{-1}(4x - 1) \) are respectively: \([3]\)
   
   (a) \(0 \leq x \leq \frac{1}{2}\) and \(-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}\)
   (b) \(-\frac{1}{2} \leq x \leq 0\) and \(-\frac{\pi}{2} \leq y \leq \frac{5\pi}{2}\)
   (c) \(0 \leq x \leq \frac{1}{2}\) and \(-\frac{\pi}{2} \leq y \leq \frac{5\pi}{2}\)
   (d) \(-\frac{1}{2} \leq x \leq 0\) and \(-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}\)
4. Given that \( \frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0 \), the unique value of \( \sin^{-1} x + \cos^{-1} x \) is \( [^4] \)

(a) C, where C is a constant
(b) 0
(c) \( \frac{\pi}{4} \)
(d) \( \frac{\pi}{2} \)

5. Which of the following graphs represents \( = \cos^{-1} 2x \)? \( [^5] \)
2. **GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS**

**Question 1**

(a) 2007 HSC Ext 1 Question 2(b)

Let \( f(x) = 2 \cos^{-1} x \)

(i) Sketch the graph of \( y = f(x) \), indicating clearly the coordinates of the endpoints of the graph.

(ii) State the range of \( f(x) \). [6]

(b) Consider the function \( f(x) = 3 \cos^{-1} \left( \frac{x}{4} \right) \)

(i) State the domain and range of the function \( f(x) \). [7]
(ii) Write down the $y$-intercept of the curve $y = f(x)$. [8]

(iii) Determine the inverse function $y = f^{-1}(x)$ and determine its domain and range. [9]

(iv) On the same coordinate plane, sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$. 
Question 2

(a) Consider the function \( f(x) = 3\sin^{-1}\left(\frac{x}{2}\right) \)

(i) Evaluate \( f(2) \). \([10]\]

(ii) State the domain and range of \( y = f(x) \). \([11]\]

(iii) Sketch the curve \( y = f(x) \)
(b) Consider the curve \( y = 3 + \tan^{-1} \sqrt{x} \)

(i) Find the domain and range of \( y = 3 + \tan^{-1} \sqrt{x} \). \([12]\)

(ii) Sketch the curve \( y = 3 + \tan^{-1} \sqrt{x} \)

(c) Consider the function \( f(x) = 2 - \sin^{-1}(3 - 2x) \)

(i) State the domain and range of the function. \([13]\)
(ii) Hence sketch the graph of \( y = f(x) \)

**Note to Students:**
Note that the negative on the \( x \) will flip the \( \sin^{-1} x \) curve about a vertical axis.

(d) (i) State the domain and range of \( y = 3\cos^{-1}(2x + 3) \). [14]

(ii) Hence sketch the graph of \( y = 3\cos^{-1}(2x + 3) \)
Question 3

Consider the function $f(x) = \sec x$.

(i) Sketch the curve $y = f(x)$ in the domain $-\pi \leq x \leq \pi$.

(ii) Explain why an inverse function $f^{-1}(x)$ does not exist. [15]

(iii) By restricting the domain of $f(x) = \sec x$ to $0 \leq x < \frac{\pi}{2}$, an inverse function $f^{-1}(x)$ exists. Sketch the graph of $y = f^{-1}(x)$.

(iv) State the range of $y = f^{-1}(x)$. [16]

(v) Show that the equation of the inverse function is $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$
3. PROPERTIES AND EXPRESSIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Question 1

(a) \( \alpha = \tan^{-1}\left(\frac{1}{2}\right) \) and \( \beta = \tan^{-1}\left(\frac{1}{3}\right) \), find the exact value of \( \tan(\alpha + \beta) \). \([17]\)

(b) Find the exact value of \( \cos\left[2\tan^{-1}\left(\frac{5}{12}\right)\right] \). \([18]\)

Did you know:
When in doubt draw a triangle
(c) Without using a calculator, show that \(2 \cos^{-1} \left( \frac{3}{5} \right) = \cos^{-1} \left( -\frac{7}{25} \right)\).

(d) Find the exact value of \(\tan \left( 2 \sin^{-1} \frac{2}{3} \right)\). \([19]\)

(e) Find the exact value of \(\sin^{-1} \frac{5}{13} + \sin^{-1} \frac{12}{13}\). \([20]\)

**Note to Students:**
Let \(\theta = \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{12}{13}\) and consider \(\sin \theta\).
Question 2

(a) Prove the following:

(i) \( \sin \left( \tan^{-1} x \right) = \frac{x}{\sqrt{x^2 + 1}}. \)

(ii) \( \tan^{-1} x = \cot^{-1} \frac{1}{x} \)

(iii) \( \cos^{-1} a + \cos^{-1} b = \cos^{-1} \left[ ab - \sqrt{(1-a^2)(1-b^2)} \right], \) provided \( 0 \leq \cos^{-1} a + \cos^{-1} b \leq \pi. \)
(b) 
(i) Prove that \( \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \)

(ii) Hence find the point of intersection of the graphs \( y = \sin^{-1}x \) and \( y = \cos^{-1}x \). [21]

(c) Express \( \cot(\sin^{-1}x) \) in terms of \( x \). [22]
Question 3 2009 HSC Mathematics Ext 1 Question 7(b) (i)

A billboard of height $a$ metres is mounted on the side of a building, with its bottom edge $h$ metres above street level. The billboard subtends an angle $\theta$ at the point $P$, $x$ metres from the building.

Use the identity $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ to show that $\theta = \tan^{-1} \left[ \frac{ax}{x^2 + h(a + h)} \right]$
Question 4  2007 HSC Mathematics Ext 1 Question 5(c)

Find the exact values of $x$ and $y$ which satisfy the simultaneous equations. [23]

$$\sin^{-1} x + \frac{1}{2} \cos^{-1} y = \frac{\pi}{3} \text{ and }$$

$$3 \sin^{-1} x - \frac{1}{2} \cos^{-1} y = \frac{2\pi}{3}$$