YEAR 12
MATHS ADV

LESSON 5:  EXPONENTIAL FUNCTIONS 1
1. OVERVIEW OF EXPONENTIAL FUNCTIONS

- Introduction to Exponential Functions

- You have been sketching and using exponential functions such as $y = 2^x$ and $y = 5^x$ for some time. We now focus on a specific exponential function $y = e^x$ where the real number $e$ is approximately $2.718281828$. Why narrow our attention down to such a specific example? The exponential function $y = e^x$ has a property which is absolutely unique. It is equal to its own derivative! That is $\frac{d}{dx}(e^x) = e^x$.

- No other function behaves like this! As a general rule the processes of calculus have a huge impact on functions. For example the cubic function $y = x^3$ is transformed into the quadratic $y' = 3x^2$ by differentiation. All functions are dramatically modified by integration and differentiation, all functions except $y = e^x$ which is miraculously immune to calculus. This simple property makes $y = e^x$, without any doubt, the most important function in all of mathematics.

- In this lesson we will define $y = e^x$ and carefully consider the usual applications and features of its derivative. Keep in mind that all of the old rules of calculus, the product, quotient and chain rules still apply to this new function. Indeed they hold for any function! This lesson will focus on the derivative of the exponential function. Lesson 6 will complete the story by considering the integral of the exponential function.
2. EXPONENTIAL FUNCTIONS

- **Definition**
  - An exponential function is a function of the form $f(x) = a^x$, where $a$ is a positive constant and $a \neq 0$

- **Features of Exponential Function**
  - The graphs of $y = a^x$ are shown in figure below for various values of the base $a$.

- When $x = 0$, $a^x = \ldots \ldots \ldots$
- All of these graphs pass through the same point $(0, \ldots \ldots \ldots)$.
- As the base $a$ gets larger, the exponential function increases more rapidly for $x > 0$.
- The graph of $y = a^x$ does not touch the $x$-axis.
How the Exponential Graph Compares with the Parabola

- The figure above shows how the exponential function $y = 2^x$ compares with the parabola $y = x^2$

- The graphs intersect three times, but ultimately the exponential curve $y = 2^x$ grows far more rapidly than the parabola $y = x^2$
The Value of $a$

The value of $a$ affects the graph of $y = a^x$ in the following ways:

1. $0 < a < 1$
   - When $0 < a < 1$, the exponential function is usually written in the form $y = a^{-x}$ since $a$ is a fraction.
   - This exponential function decreases.
   - $x \to \infty, y \to 0$
   - $x \to -\infty, y \to \infty$

![Graph of $y = a^x$, $0 < a < 1$](image.png)

**Note to Students:**
This is called exponential decay

2. $a = 1$
   - The exponential function $f(x) = 1^x$ is constant.

![Graph of $y = 1^x$](image.png)
3. \( a > 1 \)

- The exponential function \( f(x) = a^x \) increases as \( x \) increases.

- \( x \to \infty, y \to \infty \)
- \( x \to -\infty, y \to 0 \)

Note to Students:
This is called exponential growth
Concept Check 2.1

(i) Find \( \lim_{x \to \infty} (2^{-x} - 1) \). [1]

(ii) Sketch the graph of the function \( y = 2^{-x} - 1 \), clearly showing the \( y \)-intercept.

Note to Students:
First consider the sketch of \( 2^{-x} \)
Concept Check 2.2

Sketch the following graphs

(i) \( y = 2^x + 1 \)

Note to Students:
First consider the graph of \( 2^x \)

(ii) \( y = 2^{x+1} \)
(iii) \[ y = 3^{-x} \]

(iv) \[ y = -3^{-x} \]
(v) \[ y = 3 - 2^x \]

(vii) \[ y = 4^{|x|} \]
3. **INDEX LAWS**

- **Revision of Index Laws**
  - \( x^m \times x = x^{m+n} \)
  - \( x^m \div x^n = x^{m-n} \)
  - \( (x^m)^n = x^{mn} \)
  - \( x^{-m} = \frac{1}{x^m} \)
  - \( x^0 = 1 \)
  - \( x^{\frac{m}{n}} = \sqrt[n]{x^m} \)

**Concept Check 3.1**

Use index laws to simplify the following:

(a) \( \left( \frac{27}{64} \right)^{\frac{2}{3}} \)

**Note to Students:**
The denominator is the root, the numerator is the power.

(b) \( \left( \frac{2}{5} \right)^{-3} \)
(c) \[ 16^{\frac{3}{2}} \left( \frac{2}{8} \right)^{\frac{1}{3}} \]. [4]

(d) \[ 5^{2x-3} \times 5^{6-x} \]. [5]

(e) \[ \frac{3}{3^{4-x}} \]. [6]

(f) \[ \left( \frac{2^{2x}}{6^{2x}} \right)^{3} \]. [7]

(g) \[ \sqrt{2^{2x} \cdot 6^{3y}} \]. [8]
4. **EXPONENTIAL EQUATIONS**

- When solving exponential equations, consider the following equalities
  - If $a^x = a^y$ then $x = y$
  - If $a^x = b^y$ then $a = b$

Concept Check 4.1

Solve the following equations for $x$:

(a) $5^{3x-4} = 5^{3x-6}$. [9]

(b) $4^{x-6} = 2^{x+3}$. [10]

Note to Students:
Rewrite in terms of a common base

(c) $8^{3x+1} = 1$. [11]
(d) \[ \frac{3}{x^2} = 32 \] \hspace{1cm} [12]

Note to Students:
Rewrite in terms of powers of 2

(e) \[ \left(\frac{1}{4}\right)^{x+2} = 16^x \times \sqrt{8} \] \hspace{1cm} [13]
Concept Check 4.2

Solve the following equations for $x$:

(a) $(2x - 3)3^{-x} = 0$ \(^{[14]}\)

(b) $2^x - 2x(2^x) = 0$ \(^{[15]}\)

(c) $4^{2x} - 4^x - 2 = 0$ \(^{[16]}\)
5. DERIVATIVE OF EXPONENTIAL FUNCTION

- The Derivative of $2^x$

- The curve $y = 2^x$ is shown in the sketch below.

- Compute the derivative of the exponential function $f(x) = 2^x$ using the definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

\[
\begin{align*}
    f(x) &= 2^x \\
    f(x+h) &= 2^{x+h} \\
    f'(x) &= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \\
    &= \ldots \\
    &= \ldots \\
    &= \ldots
\end{align*}
\]
We found that \( f'(x) = 2^x \lim_{h \to 0} \frac{2^h - 1}{h} \).

Therefore \( f'(x) \) exists if and only if \( \lim_{h \to 0} \frac{2^h - 1}{h} \) exists.

When \( x = 0 \), \( f'(x) = 2^x \lim_{h \to 0} \frac{2^h - 1}{h} \) becomes:

\[
 f'(0) = 2^0 \lim_{h \to 0} \frac{2^h - 1}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}
\]

Hence \( \lim_{h \to 0} \frac{2^h - 1}{h} \) represents the value of the derivative of \( 2^x \) at \( x = 0 \).

We can approximate the value of \( \lim_{h \to 0} \frac{2^h - 1}{h} \) by finding the value of \( \frac{2^h - 1}{h} \) as \( h \) gets closer to zero.

To find the value of \( \frac{2^h - 1}{h} \) as \( h \) gets closer to zero, complete the following table to find an approximate value for \( \lim_{h \to 0} \frac{2^h - 1}{h} \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>1</th>
<th>0.1</th>
<th>0.001</th>
<th>0.0001</th>
<th>0.00001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2^h - 1}{h} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hence \( \lim_{h \to 0} \frac{2^h - 1}{h} \approx \) ………………………………

Use your calculator to find the value of \( \ln 2 \)

How does this compare with \( \lim_{h \to 0} \frac{2^h - 1}{h} \)?

Complete the following:

\[
 \text{Hence } \frac{d}{dx}(2^x) = \text{………………………………………}
\]

The value of the derivative of \( 2^x \) at the point (0,1) is ………………………………

Hence the equation of the tangent to the curve \( y = 2^x \) at the point (0,1) is:

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………

…………………………………………………………………………………………
The Derivative of $a^x$

- The curve $y = a^x$ is sketched below.

- Compute the derivative of the exponential function $f(x) = a^x$ using the definition of the derivative:

\[ f(x) = \ldots \]
\[ f(x + h) = \ldots \]
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \]
\[ = \ldots \]
\[ = \ldots \]
\[ = \ldots \]

- Therefore $f'(x)$ exists if and only if $\lim_{h \to 0} \frac{a^h - 1}{h}$ exists.

- When $x = 0$:

\[ f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h} \]
\[ f'(0) = a^0 \lim_{h \to 0} \frac{a^h - 1}{h} \]
\[ = \lim_{h \to 0} \frac{a^h - 1}{h} \]

- Hence $\lim_{h \to 0} \frac{a^h - 1}{h}$ represents the value of the derivative of $a^x$ at $x = 0$. 
The derivative of $a^x$ where $a$ is a constant:

\[
\frac{d}{dx}(a^x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \ln a
\]

since \(\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a\)

\[
\frac{d}{dx}(a^x) = a^x \ln a
\]
The Exponential Function $e^x$

- Since $\frac{d}{dx}(a^x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$, a new function $e^x$ is defined such that:
  $$\frac{d}{dx}(e^x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}, \quad \text{where} \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

- When $a = 2$, $\lim_{h \to 0} \frac{a^h - 1}{h} \approx \ldots \ldots \ldots \ldots$

- When $a = 3$, $\lim_{h \to 0} \frac{a^h - 1}{h} \approx \ldots \ldots \ldots \ldots$

- Hence the value of $e$ must lie between $\ldots \ldots \ldots \ldots$ and $\ldots \ldots \ldots \ldots$
  but closer to $\ldots \ldots \ldots \ldots$ since $\lim_{h \to 0} \frac{e^h - 1}{h}$.

The Derivative of $e^x$ and $ke^x$

- Investigate the value of $e$ by completing the table below.

<table>
<thead>
<tr>
<th>Approximate value of $e$</th>
<th>$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 2.5$</td>
<td></td>
</tr>
<tr>
<td>$a = 2.6$</td>
<td></td>
</tr>
<tr>
<td>$a = 2.7$</td>
<td></td>
</tr>
<tr>
<td>$a = 2.8$</td>
<td></td>
</tr>
</tbody>
</table>

- Since $e$ is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, an approximate value of $e$ is $\ldots \ldots$

- The derivative of $e^x$:
  $$\frac{d}{dx}(e^x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$
  $$= e^x \times 1$$
  $$= e^x$$

$$\frac{d}{dx}(e^x) = e^x$$
The exponential function \( f(x) = e^x \) has the property that it is its own derivative.

The geometrical significance of this fact is that the slope of a tangent to the curve \( y = e^x \) is equal to the \( y \) – coordinate of the point of contact on the curve.

The derivative of \( ke^x \) where \( k \) is a constant:

\[
\frac{d}{dx}(ke^x) = ke^x
\]

Concept Check 5.1

The diagram shows the graph of \( y = e^x \). Complete the following:

- \( y = e^x \) then \( \frac{dy}{dx} = \) …………………………………………………
- Hence at \( x = 0 \), \( \frac{dy}{dx} = \) …………………………………………………
- Therefore the gradient of the tangent at the point \( (0,1) \) on the curve \( y = e^x \) is ………
- The gradient of the tangent to the curve \( y = e^x \) at any point \( (x, y) \) is given by

\[
\frac{dy}{dx} = e^x
\]

Did You Know:

\( e \) is just a special number like \( \pi \). Its approximate value is 2.718. Your calculator will have an \( e^x \) button.

Discussion Question:

Write down the exact value of \( e \). \(^{[17]}\)
Derivative of $e^{f(x)}$

- To differentiate $y = e^{f(x)}$ let $u = f(x)$
  \[ y = \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

- Here, $\frac{dy}{du} = \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

  Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

  where $\frac{du}{dx} = \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

- Hence

  \[ \frac{dy}{dx} = f'(x) e^u \]

  \[ \therefore \frac{dy}{dx} = f'(x) e^{f(x)} \]

\[ \frac{d}{dx}(e^{f(x)}) = f'(x) e^{f(x)} \]

Did you know:
The simple fact that $\frac{d}{dx}(e^x) = e^x$ makes $y = e^x$ the most important function in mathematics.
Concept Check 5.2

Differentiate the following exponential functions with respect to $x$.

(i) $\frac{d}{dx} (2e^x)$  \[18\]

(ii) $\frac{d}{dx} (e^{5-2x})$  \[19\]

(iii) $\frac{d}{dx} (e^{x^2-2x})$  \[20\]

(iv) $\frac{d}{dx} \left( \frac{e^x - e^{-2x}}{4} \right)$  \[21\]

(v) $\frac{d}{dx} \left( e^{\frac{1}{x}} \right)$  \[22\]
(vi) \( \frac{d}{dx} \left( \frac{1}{e^{x^2}} \right) \) \[23\]

Discussion Question:
What is the derivative of \( y = e^{4x} \)? \[24\]

(vii) \( \frac{d}{dx} \left( 5e^{7-2x} + 4ex \right) \) \[25\]

(viii) \( \frac{d}{dx} \left( 5x^3 - \frac{2}{x} + \frac{x^2}{4} \right) \) \[26\]

Discussion Question:
Evaluate
(a) \( \lim_{x \to \infty} \frac{x^2}{e^x} \)
(b) \( \lim_{x \to \infty} \frac{x^{20000000}}{e^{0.0000001x}} \)
Rules for Differentiation

(a) Chain Rule: \[ \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x) \]

(b) Product Rule:
\[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

(c) Quotient Rule:
\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

Concept Check 5.3

Use the chain rule to differentiate the following:

(a) \[ \frac{d}{dx}(e^{2x} - 4x)^3 \]

Note to Students:
Differentiate first what you evaluate last. The last calculation here is cubing, so this is the first concern under differentiation

(b) \[ \frac{d}{dx}\sqrt{4 - 3e^x} \]
(c) \[ \frac{d}{dx} \left( \frac{4}{e^{3x^2} + 5x^2 - 3} \right) \] \[ \text{[29]} \]

(d) \[ \frac{d}{dx} \left( \frac{1}{\sqrt{e^x - e^{-2x}}} \right) \] \[ \text{[30]} \]

(e) \[ \frac{d}{dx} \left( 3e^{4x} + 3x^2 \right)^{\frac{1}{4}} \] \[ \text{[31]} \]
Concept Check 5.4

Use the product rule to differentiate the following:

Did you know:

\[(uv)' = u'v + v'u\]

(a) \[\frac{d}{dx}(2xe^x)\]

(b) \[\frac{d}{dx}\left((3x + 5)e^{2x-7}\right)\]

(c) \[\frac{d}{dx}\left(\sqrt{x}e^{3-7x^2}\right)\]
(d) \[ \frac{d}{dx}(xe^{-x^2}) \] \[ ^{[35]} \]

(e) \[ \frac{d}{dx}\left[(3x^2 + 5x - 2)e^{\frac{4}{x^2}}\right] \] \[ ^{[36]} \]
Concept Check 5.5

Use the quotient rule to differentiate the following

Did You Know:

\[
\left( \frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}
\]

(a) \[ \frac{d}{dx} \left( \frac{e^{-x^2}}{x} \right) \] \[ \text{[37]} \]

(b) \[ \frac{d}{dx} \left( \frac{e^{3x}}{1 + e^x} \right) \] \[ \text{[38]} \]

(c) \[ \frac{d}{dx} \left( \frac{e^{-x^2}}{3x - 4} \right) \] \[ \text{[39]} \]
(d) \[
\frac{d}{dx} \left[ \frac{e^{2x} - e^{-2x}}{e^{3x} + e^{-2x}} \right] \quad \text{[40]}
\]

(e) \[
\frac{d}{dx} \left[ \frac{2x^2 + 3}{e^{3x}} \right] \quad \text{[41]}
\]

(f) \[
\frac{d}{dx} \left[ \frac{e^{4x} + 3}{e^{-4x} - 3} \right] \quad \text{[42]}
\]
Concept Check 5.6

(i) Find the derivative of \( y = e^{3x} \) \([43]\) 

(ii) Hence find the equation of the tangent to the curve \( y = e^{3x} \) at the point \((0, 1)\) \([44]\)

Concept Check 5.7

(i) Show that the function \( y = e^{2x} + e^{-3x} \) satisfies the differential equation \( y'' + y' - 6y = 0 \)

(ii) Show that the function \( y = Ae^{-x} + Bxe^{-x} \) satisfies the differential equation \( y'' + 2y' + y = 0 \)
Concept Check 5.8

(a) For what values of $a$ does the function $y = ae^{ax}$ satisfy the equation $y'' + y' - 6y = 0$? \[45\]

(b) Find the values of $\lambda$ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$. \[46\]

Concept Check 5.9

If $f(x) = e^{-2x}$,

(a) Find the following derivatives:

(i) $f''(x)$ \[47\]

(ii) $f'''(x)$ \[48\]

(iii) $f''''(x)$ \[49\]

(b) Hence find $f^{(6)}(x)$ \[50\]
6. APPLICATIONS TO CURVE SKETCHING

Revision of Curve Sketching

- When \(\frac{dy}{dx} > 0\) the curve increases or rises.
- When \(\frac{dy}{dx} < 0\) the curve decreases or falls.
- Stationary points occur where \(\frac{dy}{dx} = 0\).
- When \(\frac{d^2y}{dx^2} > 0\) the concavity of the curve is upward.
- When \(\frac{d^2y}{dx^2} < 0\) the concavity of the curve is downward.
- When \(\frac{d^2y}{dx^2} = 0\) a possible point of inflexion may occur at this value of \(x\). For a point of inflexion to occur, the concavity of the curve must change about this value of \(x\).
- For a relative minima at \(x = x_0\), \(\frac{dy}{dx} = 0\) and \(\frac{d^2y}{dx^2} > 0\) at \(x = x_0\).
- For a relative maxima at \(x = x_0\), \(\frac{dy}{dx} = 0\) and \(\frac{d^2y}{dx^2} < 0\) at \(x = x_0\).
- A horizontal point of inflexion occurs at \(x = x_0\) if \(\frac{dy}{dx} = 0\), \(\frac{d^2y}{dx^2} = 0\) and a change in concavity occurs about \(x = x_0\).
Concept Check 6.2

Consider the curve $y = xe^x$.

(i) Find the coordinates of the stationary point and determine their nature. [51]

Note to Students:
Remember that $e^x \neq 0$, so you are free to divide both sides of an equation by $e^x$.

(ii) Find the coordinates of any points of inflexion. [52]

(iii) For what values of $x$ does the curve increase with upward concavity? [53]
(iv) Discuss the behaviour of the curve as $x \to \pm \infty$. \[54\]

**Note to Students:**
As $x \to -\infty$, $e^{-x} \to 0$. But the exponential function has more power than the '$x$', so
\[\lim_{x \to -\infty} xe^{-x} = -\infty \times 0 = 0\]

(v) Sketch the curve
7. PAST HSC QUESTIONS

Did you know:
Questions involving $e^x$ will appear throughout the calculus based papers. In advanced maths we have simple integrals and derivatives together with the theory of exponential growth and decay. In the Extension 1 course the exponential function will appear in Newton’s law of cooling questions and also in general calculus problems.

2009 H.S.C Mathematics Q2(a)(ii)

Differentiate with respect to $x$:

\[ \frac{d}{dx}(e^x + 1)^2 \]

\[ = 2(e^x + 1) \cdot e^x \]

\[ = 2e^{2x} + 2e^x \]

NOTES FROM THE MARKING CENTRE

The chain rule was the most popular method used for differentiation with only a small number of candidates choosing to expand the original expression. Common incorrect responses included $f'(x) = 2(e^x + 1), 2(e^x + 1)^2 \cdot e^x$ and $(e^x + 1) \times e^x$. Those candidates who chose the expansion method quite often incorrectly expanded, obtaining $f(x) = e^{2x} + 2e^x + 1$. 
2011 H.S.C Mathematics Q2(d)

Find the derivative of \( y = x^2 e^x \) with respect to \( x \) \[^{[56]}\]

NOTES FROM THE MARKING CENTRE

This part of the question was completed successfully by most candidates. Many were assisted by using an organisation area for \( u, u', v \) and \( v' \). Few quoted the product rule but most were able to write the derivative expressions correctly. The occasional errors were in having an inappropriate negative sign in the rule, or by having the derivative of \( e^x \) as \( xe^x \).

2006 H.S.C Mathematics Extension 1 Q5(a)

Show that \( y = 10 e^{-0.7t} + 3 \) is a solution of \( \frac{dy}{dt} = -0.7 \ (y - 3) \).

NOTES FROM THE MARKING CENTRE

Most candidates were able to score two marks on this part. The most popular approach was to find \( \frac{dy}{dt} \) and this enabled candidates to gain full marks with little difficulty. Candidates who tried to integrate \( t = \int \frac{1}{0.7(y - 3)} \) \( dy \) were not only slowed down but also made mistakes in dealing with the constant. A few candidates quoted the formulae: \( \frac{dN}{dt} = k (N - N_0), \ N = N_0 + Ae^{kt} \). Some candidates stated that since the given equations were in this form then one must be a solution to the other. These candidates were expected to ‘show’ that this was true.