YEAR 11
MATHS ADV

LESSON 6: TRIGONOMETRIC RATIOS 1
1. REVIEW OF TRIGONOMETRIC RATIOS

- Trigonometric Ratios
  - In any right angled triangle:

\[
\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}
\]

\[
\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}
\]

\[
\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}
\]

Did You Know:

An easy way to remember these results is “SOH CAH TOA”

- **SOH**: \( \sin \theta \) = opposite/hypotenuse
- **CAH**: \( \cos \theta \) = adjacent/hypotenuse
- **TOA**: \( \tan \theta \) = opposite/adjacent
cosec $\theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite side}}$

$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent side}}$

$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent side}}{\text{opposite side}}$

**Did You Know:**
The formula for cosec $\theta$, sec $\theta$ and cot $\theta$ are easily remembered by using the third letter:
- $\text{coSec} \theta = \frac{1}{\sin \theta}$
- $\text{seC} \theta = \frac{1}{\cos \theta}$
- $\text{coT} \theta = \frac{1}{\tan \theta}$
Concept Check 1.1

(a) In the diagram shown:

(i) Determine the missing side using Pythagoras’ Theorem. [1]

(ii) Write down the values of:

\[ \sin \theta = \] [2]

\[ \cos \theta = \] [3]

\[ \tan \theta = \] [4]

(b) Find the value of the \( x \) in the following diagrams giving your answers for lengths correct to 1 decimal place and for angles to the nearest minute. All measurements are in cm. [5]
a) 

\[ \angle \text{33°9'} \]

12.4

\[ x \]

b) 

\[ \angle \text{19.4} \]

12.6

\[ x \]
Exact Values

Using the triangles below, complete the table.

<table>
<thead>
<tr>
<th></th>
<th>$\theta = 30^\circ$</th>
<th>$\theta = 45^\circ$</th>
<th>$\theta = 60^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>[8]</td>
<td>[9]</td>
<td>[10]</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>[11]</td>
<td>[12]</td>
<td>[13]</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>[14]</td>
<td>[15]</td>
<td>[16]</td>
</tr>
</tbody>
</table>

Note to Students:
Trigonometric ratios of $30^\circ$, $45^\circ$ and $60^\circ$ should always be presented in exact form.
2. **ANGLES OF ANY MAGNITUDE**

**Note to Students:**
There are many different approaches to the evaluation of the trigonometric ratios of angles larger than 90°. Make sure that you can quickly and efficiently deal with these problems.

- **Trigonometric Ratios in the First Quadrant**
  - The trigonometric functions sine and cosine are defined in terms of the coordinates of points lying on the unit circle $x^2 + y^2 = 1$.
Consider the diagram which shows a right-angled triangle $AOP$ within a unit circle.

Using the triangle $AOP$, show that $x = \cos \theta$ and $y = \sin \theta$.

Is $\cos \theta$ (x-coordinate of P) in the first quadrant positive or negative? [17]

Is $\sin \theta$ (y-coordinate of P) in the first quadrant positive or negative? [18]

Is $\tan \theta$ in the first quadrant positive or negative? [19]

Hence, all the trigonometric ratios are positive in the first quadrant.
Concept Check 2.1

Without using a calculator, find the exact value of the following:

(a) \( \sin 60^\circ \cos 45^\circ + \cos 45^\circ \sin 60^\circ \) \[20\]

(b) \( \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \) \[21\]

(c) \( \sec^2 60^\circ - \tan^2 60^\circ \) \[22\]

(d) \( \frac{\sin^2 30^\circ + \cos^2 30^\circ}{2 \sin 60^\circ \cos 60^\circ} \) \[23\]

(e) \( 4 \sin^3 60^\circ - \sin 60^\circ \) \[24\]
Concept Check 2.2

Without using a calculator, show the following are true.

(a) \[ \frac{\tan 30^\circ}{\sin 30^\circ} = \sec 30^\circ \]

(b) \[ \cos^2 30^\circ + \sin^2 30^\circ = 1 \]

(c) \[ 1 + \cot^2 60^\circ = \csc^2 60^\circ \]

(d) \[ \frac{1 - \tan^2 30^\circ}{2 \tan 30^\circ} = \cot 60^\circ \]
Trigonometric Ratios in the Second Quadrant

- Consider the right angle triangle in the second quadrant.

![Diagram of a right angle triangle in the second quadrant with coordinates Q(-x, y) and P(x, y).]

Is \( \cos \theta \) (x-coordinate of Q) in the second quadrant positive or negative?\(^{[25]}\)

Is \( \sin \theta \) (y-coordinate of Q) in the second quadrant positive or negative?\(^{[26]}\)

Is \( \tan \theta \) in the second quadrant positive or negative?\(^{[27]}\)

Hence the positive trigonometric ratio in the second quadrant is \( \sin \theta \)

- From the diagram, we can see that the coordinates of Q and P are related by symmetry:
  - The coordinates of P: \( (x, y) = (\cos \theta, \sin \theta) \)
  - The coordinates of Q: \( (-x, y) = [\cos(180 - \theta), \sin(180 - \theta)] \)

  Hence, using the triangle in the second quadrant:

  \[
  \sin(180 - \theta) = y = \sin \theta \\
  \cos(180 - \theta) = -x = -\cos \theta \\
  \tan(180 - \theta) = \frac{y}{-x} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta
  \]
Trigonometric Ratios in the Third Quadrant

- Consider the right angle triangle in the third quadrant.

Is \( \cos \theta \) (x-coordinate of Q) in the third quadrant positive or negative? \(^{28}\)

Is \( \sin \theta \) (y-coordinate of Q) in the third quadrant positive or negative? \(^{29}\)

Is \( \tan \theta \) in the third quadrant positive or negative? \(^{30}\)

Hence the positive trigonometric ratio in the third quadrant is \( \tan \theta \)

- From the diagram, we can see that the coordinates of Q and P are related by symmetry:

The coordinates of P: \((x, y) = (\cos \theta, \sin \theta)\)

The coordinates of Q: \((-x, -y) = [\cos(180 + \theta), \sin(180 + \theta)]\)

Hence, using the triangle in the third quadrant:

\[
\sin(180 + \theta) = -y = -\sin \theta \\
\cos(180 + \theta) = -x = -\cos \theta \\
\tan(180 + \theta) = \frac{-y}{-x} = \frac{\sin \theta}{\cos \theta} = \tan \theta
\]
Trigonometric Ratios in the Fourth Quadrant

Consider the right angle triangle in the **fourth quadrant**

![Diagram of a right angle triangle in the fourth quadrant]

Is \( \cos \theta \) (x-coordinate of Q) in the fourth quadrant positive or negative?  

\[ \text{Is } \cos \theta \text{ (x-coordinate of Q) in the fourth quadrant positive or negative?} \]

\[ \text{Is } \sin \theta \text{ (y-coordinate of Q) in the fourth quadrant positive or negative?} \]

\[ \text{Is } \tan \theta \text{ in the fourth quadrant positive or negative?}\]

**Hence the positive trigonometric ratio in the fourth quadrant is** \( \cos \theta \)

From the diagram, we can see that the coordinates of Q and P are related by symmetry:

The coordinates of P: \((x, y) = (\cos \theta, \sin \theta)\)

The coordinates of Q: \((x, -y) = [\cos(360 - \theta), \sin(360 - \theta)]\)

Hence, using the triangle in the fourth quadrant:

\[
\sin(180 + \theta) = -y = -\sin \theta \\
\cos(180 + \theta) = x = \cos \theta \\
\tan(180 + \theta) = \frac{-y}{x} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta
\]
Negative Angles

Consider the right angle triangle in the **fourth quadrant**

A negative angle lies in the fourth quadrant.

**Hence the positive trigonometric ratio for negative angles is** $\cos \theta$

From the diagram, we can see that the coordinates of $Q$ and $P$ are related by symmetry:

The coordinates of $P$: $(x, y) = (\cos \theta, \sin \theta)$

The coordinates of $Q$: $(x, -y) = [\cos(-\theta), \sin(-\theta)]$

Hence, using the triangle in the fourth quadrant:

\[
\sin(-\theta) = -y = -\sin \theta
\]

\[
\cos(-\theta) = x = \cos \theta
\]

\[
\tan(-\theta) = \frac{-y}{x} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta
\]
Angles Greater than 360°

Recall that adding (or subtracting) 360° is equivalent to a full revolution around the unit circle. Thus, you will return to the same quadrant in the same position.
Positive Ratios

- The positive ratios can be summarized as:

- You can use the following phrases to help you remember the positive ratios:
  - All Stations To Central
  - All Surfers To Coogee
  - Add Sugar To Coffee
Formulas for General Angles

- Given that $\theta$ is an acute angle, then the following are equivalent ratios:

$$\sin(180^\circ - \theta) = \sin \theta$$
$$\cos(180^\circ - \theta) = -\cos \theta$$
$$\tan(180^\circ - \theta) = -\tan \theta$$

$$\sin(180^\circ + \theta) = -\sin \theta$$
$$\cos(180^\circ + \theta) = -\cos \theta$$
$$\tan(180^\circ + \theta) = \tan \theta$$

$$\sin(360^\circ - \theta) = -\sin \theta$$
$$\cos(360^\circ - \theta) = \cos \theta$$
$$\tan(360^\circ - \theta) = -\tan \theta$$

$$\sin(-\theta) = -\sin \theta$$
$$\cos(-\theta) = \cos \theta$$
$$\tan(-\theta) = -\tan \theta$$
Concept Check 2.3

Find the exact values of the following.

Did You Know:
When faced with the trigonometric ratios of a large angle we have a two stage attack:
1. Make the big angle acute by moving to the related angle $\angle$, which is just the angle up or down off the horizontal (in the unit circle).
2. Use the ASTC diagram to determine the sign.
If the related angle is $\angle = 30^\circ$, $45^\circ$ or $60^\circ$, present your answer in exact form.

(i) $\sin 210^\circ$ [34]

- **Step 1:** Identify the quadrant that the angle lies in.

- **Step 2:** Find the acute angle, $\theta$.
  - $2^{nd}$ quadrant $\rightarrow (180 - \theta)$
  - $3^{rd}$ quadrant $\rightarrow (180 + \theta)$
  - $4^{th}$ quadrant $\rightarrow (360 - \theta)$

- **Step 3:** Determine whether your trigonometric ratio is positive or negative in your quadrant (hint: use ASTC). Then, write down the relevant trigonometric ratio in terms of your acute angle.

- **Step 4:** Find the exact value of your trigonometric ratio.
(ii) \( \tan(-315°) \) \[35\]

**Note to Students:**
We make very big angles small by subtracting 360° (one complete revolution) and negative angles positive by adding 360°.

- **Step 1:** Identify the quadrant that the angle lies in.

- **Step 2:** Find the acute angle. (Hint: add 360°)

- **Step 3:** Determine whether your trigonometric ratio is positive or negative in your quadrant (hint: use ASTC). Then, write down the relevant trigonometric ratio in terms of your acute angle.

- **Step 4:** Find the exact value of your trigonometric ratio.

(iii) \( \csc 240° \) \[36\]

**Note to Students:**
Never think about \( \csc, \sec \) or \( \cot \). Always move immediately to \( \sin, \cos \) or \( \tan \).
(iv) \(\cos 135^\circ\) [37]

(v) \(\sec 420^\circ\) [38]

(vi) \(\cot(-510^\circ)\) [39]

Note to Students:
Add 360°… twice!

(vii) \(\sin 585^\circ\) [40]
Concept Check 2.4

Express the following ratios as the ratio of an acute angle.

(a) \( \cos 154^\circ \)  

- Identify the quadrant that the angle lies in.

- Step 2: Find the acute angle, \( \theta \).
  
  2\(^{\text{nd}}\) quadrant \( \rightarrow (180 - \theta) \)
  3\(^{\text{rd}}\) quadrant \( \rightarrow (180 + \theta) \)
  4\(^{\text{th}}\) quadrant \( \rightarrow (360 - \theta) \)

- Step 3: Determine whether your trigonometric ratio is positive or negative in your quadrant (hint: use ASTC). Then, write down the relevant trigonometric ratio in terms of your acute angle.

(b) \( \csc 276^\circ 19' \) (Hint: \( \csc x = \frac{1}{\sin x} \))

(c) \( \tan 275^\circ \)
(d) \( \sec 328^0 \) \[44\]

(e) \( \sin 205^\circ 8' \) \[45\]

(f) \( \cos 290^\circ 42' \) \[46\]

(g) \( \sec 256^\circ 54' \) \[47\]
Concept Check 2.5

Express each of the following in terms of $\theta$ only and simplify where possible.

(i) \[
\frac{\sin (180^\circ - \theta)}{\sin (360^\circ + \theta)} \quad [48]
\]

**Note to Students:**
Adding or subtracting $360^\circ$ never has an impact.

(ii) \[
\frac{\cot (-\theta)}{\tan (180^\circ + \theta)} \quad [49]
\]

(iii) \[
\cos (360^\circ - \theta) - \cos (180^\circ - \theta) \quad [50]
\]
Complementary Angles

Given that $\theta$ is an acute angle, then the following are equivalent ratios:

\[
\sin(90 - \theta) = \cos \theta \quad \cos(90 - \theta) = \sin \theta \quad \tan(90 - \theta) = \cot \theta
\]
\[
\cot(90 - \theta) = \tan \theta \quad \sec(90 - \theta) = \csc \theta \quad \csc(90 - \theta) = \sec \theta
\]

Note to Students:
Treat “co” as if it stands for “complement of”. The third letter then indicates the trigonometric ratio that is complementary.

- $\cos \theta \rightarrow$ complement of $\cos \theta$ is $\sin \theta$
- $\cot \theta \rightarrow$ complement of $\cot \theta$ is $\tan \theta$
- $\csc \theta \rightarrow$ complement of $\csc \theta$ is $\sec \theta$

Concept Check 2.6

Fill in the missing spaces.

(i) $\cos 30^\circ = \sin \ldots\ldots\ldots\ldots\ldots\ldots\ldots[51]$
(ii) $\sin 25^\circ = \cos \ldots\ldots\ldots\ldots\ldots\ldots\ldots[52]$
(iii) $\sec 60^\circ = \cos ec.\ldots\ldots\ldots\ldots\ldots\ldots\ldots[53]$
(iv) $\cos ec45^\circ = \sec \ldots\ldots\ldots\ldots\ldots\ldots\ldots[54]$
(v) $\tan 10^\circ = \cot \ldots\ldots\ldots\ldots\ldots\ldots\ldots[55]$
(vi) $\cot 73^\circ = \tan \ldots\ldots\ldots\ldots\ldots\ldots\ldots[56]$
Concept Check 2.7

Simplify the following without using a calculator:

(a) \( \frac{\sin 20^\circ}{\cos 70^\circ} \) [57] (Hint: make the top and the bottom into the same trigonometric ratio)

(b) \( \sin 15^\circ + \cos 75^\circ \) [58]

(c) \( \frac{\tan 70^\circ}{\cot 20^\circ} \) [59]
Concept Check 2.8

If \( \tan x = \frac{1}{3} \) and \( x \) is acute, write down the exact values of:

(i) \( \tan(90^\circ - x) \) \([60]\)

(ii) \( \cos(90^\circ + x) \) \([61]\)

Step 1: Identify the quadrant that the angle lies in.

Step 2: Find the acute angle.
- 2nd quadrant \( \rightarrow (180 - \theta) \)
- 3rd quadrant \( \rightarrow (180 + \theta) \)
- 4th quadrant \( \rightarrow (360 - \theta) \)

\[ 90^\circ + x = 180^\circ - \theta \]

Step 3: Write down the relevant trigonometric ratio in terms of your acute angle.

Step 4: Determine the exact value of your trigonometric ratio using a right angle triangle and \( \tan x = \frac{1}{3} \).
(iii) \( \sin(270^\circ + x) \) \(^{[62]}\)

Step 1: Identify the quadrant that the angle lies in.

Step 2: Find the acute angle.
- 2nd quadrant \( \rightarrow (180 - \theta) \)
- 3rd quadrant \( \rightarrow (180 + \theta) \)
- 4th quadrant \( \rightarrow (360 - \theta) \)

\[ 270^\circ + x = 360^\circ - \theta \]

Step 3: Write down the relevant trigonometric ratio in terms of your acute angle, \( \theta \).

Step 4: Determine the exact value of your trigonometric ratio using a right angle triangle and \( \tan x = \frac{1}{3} \).

(iv) \( \cos(270^\circ - x) \) \(^{[63]}\)
Discussion Question:

When dealing with angles off the vertical axis $90° \pm \theta, 270° \pm \theta$ we have the following short cut.
1. Use the ASTC diagram to determine the sign of the ratio.
2. Move to the complementary ratio by adding or removing “co”.

Examples:
1. $\tan(90° - x)$
   The angle $90° - x$ is in Quadrant 1 where $\tan$ is positive.
   Therefore $\tan(90° - x) = \cot x$.
2. $\cos(90° + \theta)$
   The angle $90° + \theta$ is in Quadrant 2 where $\cos$ is negative.
   Therefore $\cos(90° + \theta) = -\sin \theta$.
3. $\sin(270° + x)$
   The angle $270° + x$ is in Quadrant 4 where $\sin$ is negative.
   Therefore $\sin(270° + x) = -\cos x$.
4. $\tan(270° - x)$
   The angle $270° - x$ is in Quadrant 3 where $\tan$ is positive.
   Therefore $\tan(270° - x) = \cot x$. 
Concept Check 2.9

Find the value of $x$ for the following equations:

(i) \( \sin x = \cos 50^\circ \) \[64\]

(ii) \( \cos ecx = \sec 26^\circ \) \[65\]

(iii) \( \tan(x - 20) = \cot 55^\circ \) \[66\]

(iv) \( \cos(3x - 8) = \sin 83^\circ \) \[67\]

(v) \( \cot(2x + 7) = \tan 73^\circ \) \[68\]

(vi) \( \sec(5 - 4x) = \cos ecx 51^\circ \) \[69\]
Given One Ratio, Find the Others

Example

(i) Given \( \tan \theta = \frac{3}{4} \) and \( \sin \theta < 0 \), find the exact values of \( \cos \theta \) and \( \sin \theta \). \(^{[70]}\)

Step 1: Identify the quadrant where \( \theta \) lies.

Step 2: Complete the triangle(s) and use Pythagoras' Theorem to find the missing side

Step 3: Find the other ratios using the triangle(s).

Note to Students:
Let the triangles take care of the ratios and the ASTC diagram take care of the sign.
(ii) Given \( \tan \alpha = \frac{1}{\sqrt{7}} \) and \( 180^\circ < \alpha < 270^\circ \) and \( \tan \beta = 1 \) and \( 0^\circ < \beta < 90^\circ \), find the exact value of \( \sin \alpha + \cos \beta \). \[71\]

**Step 1:** Identify the quadrant where \( \alpha \) and \( \beta \) lie

**Step 2:** Complete the triangle(s) and use Pythagoras' Theorem to find the missing sides.

**Step 3:** Find the other ratios using the triangle(s) and answer the question.
Concept Check 2.10

(i) Given \( \cos e c \theta = \frac{8}{7} \) find the exact values of \( \tan \theta \) and \( \cos \theta \) for \( 0^\circ < \theta < 360^\circ \). [72]

(ii) Given \( \tan \theta = -\frac{3}{4} \) and \( \cos \theta > 0 \), find the exact values of \( \cos \theta \) and cosec \( \theta \). [73]
3. **LESSON REVIEW**

1. Find the value of the $x$ in the following diagrams giving your answers for lengths correct to 1 decimal place and for angles to the nearest minute. All measurements are in cm.

   a) 
   
   b) 

2. Use the right-angled triangle given below to evaluate the following:

   a) $\csc \theta = $  
   b) $\cot \theta = $  
   c) $\sec \theta = $  
   d) $\sin \theta = $  
   e) $\cos \theta = $
3. Without using a calculator, find the exact value of the following:

(a) \(\frac{\sin 30^\circ}{\cos 60^\circ}\) [74]

(b) \(1 - 2 \sin^2 45^\circ\) [75]

(c) \(\cos e^2 45^\circ - \cot^2 45^\circ\) [76]

(d) \(\frac{1 - \tan^2 30^\circ}{2 \tan 30^\circ}\) [77]

(e) \(\frac{1 - \sin 45^\circ}{1 + \sin 45^\circ}\) [78]

(f) \(\sin 45^\circ \tan 30^\circ\) [79]
4. Without using a calculator, show the following are true.

(a) \[
\frac{\tan 30^\circ}{1 - \tan^2 30^\circ} = \frac{\tan 60^\circ}{2}
\]

(b) \[2 \sin 45^\circ \cos 45^\circ = \sin 90^\circ\]

(c) \[1 - 2 \cos^2 30^\circ = \cos 60^\circ\]

(d) \[
\frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} = \tan 30^\circ
\]
5. Express the following ratios as the ratio of an acute angle.

(a) \( \sin 171\degree \) \([80]\)

(b) \( \cot 196\degree \) \([81]\)

(c) \( \sin 215\degree 8' \) \([82]\)

(d) \( \cot 112\degree 36' \) \([83]\)

(e) \( \sec 286\degree 32' \) \([84]\)

(f) \( \tan 298\degree 26' \) \([85]\)

(g) \( \csc(−387\degree 34') \) \([86]\)